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A TREATISE ON

ATTRACTIONS, LAPLACE'S FUNCTIONS,

AND THE

FIGURE OF THE EARTH.

BY

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PREFACE TO THE THIRD EDITION.

THIS edition is a complete revise of the former ones; improvements are introduced in many places, especially in the Chapters on the Figure of the Earth; and new problems are given either as illustrations of the principles developed in the work, or as bearing upon its main design. An inspection of the Table of Contents will give a comprehensive view of the various matters here taken up.

My chief design in this treatise has been to give an answer to the question, Has the Earth acquired its present Form from being originally in a fluid state? The problem is one of great interest, and involves many investigations of high importance.

Taking the Law of Universal Gravitation as the basis, in the first part of the treatise, I calculate the resultant force exerted on a point by an assemblage of particles endowed with this attracting power, and held together in the form of a sphere, homogeneous or heterogeneous, next of a spheroid, then of an irregular mass consisting of layers nearly spherical, thus approximating more and more to the case of the Earth. This investigation gives me the opportunity of introducing the remarkable analysis of Laplace, which I have endeavoured to put in a clear light, and to free from objections which have been urged against it. The first part of the treatise is closed

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with a Chapter in which is calculated the local effect on the direction of gravity caused by irregular masses at the surface of the Earth, such as exist in table lands, vast mountain regions like the Himmalayas, and hollows filled by the ocean which is of less density than rock; and also wide-spread but slight deficiencies or excesses of matter in the crust below. All these are of importance in the problem which it is my ulterior design to solve, as they furnish the means of explaining anomalies which would otherwise be unaccountable.

The second part of the treatise is occupied in calculating the Figure of the Earth, first upon the hypothesis of its being a fluid mass, and then on geodetical principles. It is shown that the mass now consists of nearly spherical strata, whatever its former history may have been; that, on the fluid hypothesis, the form of its surface and of all its internal layers must be oblate spheroids; and that the plumb-line must be everywhere a normal to its surface. The intimate connexion between the form of the surface and the internal arrangement of the mass is shown by demonstrating the converse of the above; viz. that if the form of the surface be a spheroid of equilibrium, the earth's mass must necessarily be arranged according to the fluid law, whether the mass is or has been fluid, in part or in whole, or not. This is contrary to the belief which some others have entertained. Four tests are next applied to the fluid theory of the earth; by determining what is (1) the law of gravity on the surface which it leads to, (2) the amount of perturbation in the moon's motion in latitude, (3) the amount of precession, and (4) the value of the earth's ellipticity; and by comparing these results with those of observation and experiment. Remarks on the thickness of the earth's crust conclude the first Chapter. In the second the figure is determined geodetically.

The method of Bessel at present in use for this purpose is shown to be erroneous in one particular, and is corrected so as to bring into the calculation the effect of Local Attraction. The degree of uncertainty which that disturbing element brings into the calculation of the figure of the earth is pointed out; and it is shown how, with great probability of a correct result, the ambiguity may be removed by a comparison of the three long arcs, the Anglo-Gallic, the Russian, and the Indian. I believe this is the first time that the mean figure has been calculated, the disturbing effect of local attraction being brought into the calculation throughout. After some propositions on the sea-level, on mapping countries, and on differences of local attraction in the stations of the Indian Arc, the volume is closed by a summary of the argument regarding the hypothesis of the original fluidity of the earth, an hypothesis which I consider to be established beyond doubt.

J. H. PRATT.

CALCUTTA, 1865.

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ATTRACTIONS AND LAPLACE'S FUNCTIONS.

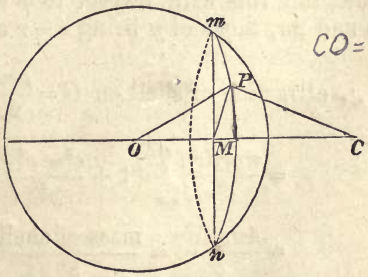
1. THE Law of Universal Gravitation teaches us, that every particle of matter in the universe attracts every other particle of matter, with a force varying directly as the mass of the attracting particle and inversely as the square of the distance between the attracted and the attracting particles. Taking this law as our basis of calculation, we shall investigate the amount of attraction exerted by spherical, spheroidal, and irregular nearly-spherical masses upon a particle, and apply our results in the second part of this Treatise to discover the Figure of the Earth. We shall also show how the attraction of irregular masses lying at the surface of the Earth may be estimated, in order afterwards to ascertain whether the irregularities of mountain-land and the ocean can have any effect on the calculation of this figure.

CHAPTER I.

ON THE ATTRACTION OF SPHERICAL AND SPHEROIDAL BODIES.

PROP. *To find the resultant attraction of an assemblage of particles constituting a homogeneous spherical shell of very small thickness upon a particle outside the shell: the law of attraction of the particles being that of the inverse square.*

2. Let O be the centre of the shell, P any particle of it, $OP=r$, dr the thickness, C the attracted particle, $\angle POC=\theta$; $mPMn$ a plane perpendicular to OC, ϕ the angle which the plane POC makes with the plane of the paper, $PC=y$.



The attraction of the whole shell evidently acts in CO.

Let OP revolve about O through a small angle $d\theta$ in the plane MOP; then $rd\theta$ is the space described by P. Again, let OPM revolve about OC through a small angle $d\phi$, then $r \sin\theta d\phi$ is the space described by P. And the thickness of the shell is dr. Hence the volume of the elementary portion of the shell thus formed at P equals $rd\theta \,.\, r \sin\theta d\phi \,.\, dr$ ultimately, since its sides are ultimately at right angles to each other.

Then, if the unit of attraction be so chosen, that it equals the attraction of the unit of mass at the unit of distance, the attraction of the elementary mass at P on C in the direction CP

$$= \frac{\rho r^2 \sin\theta \, dr d\theta d\phi}{y^2}, \quad \rho \text{ the density of the shell;}$$

$$\therefore \text{ attraction of } P \text{ on } C \text{ in } CO = \frac{\rho r^2 \sin\theta \, dr d\theta d\phi}{y^2} \frac{c - r\cos\theta}{y}.$$

We shall eliminate θ from this equation by means of

$$y^2 = c^2 + r^2 - 2cr\cos\theta,$$

$$\therefore \sin\theta \frac{d\theta}{dy} = \frac{y}{cr}, \quad c - r\cos\theta = \frac{y^2 + c^2 - r^2}{2c};$$

$$\therefore \text{ attraction of } P \text{ on } C \text{ in } CO = \frac{\rho r dr}{2c^2}\left(1 + \frac{c^2 - r^2}{y^2}\right) dy \, d\phi.$$

To obtain the attraction of all the particles of the shell we integrate this with respect to ϕ and y, the limits of ϕ being 0 and 2π, those of y being $c - r$ and $c + r$;

$$\therefore \text{ attraction of shell on } C = \frac{\rho r dr}{2c^2} \int_{c-r}^{c+r} \int_0^{2\pi} \left(1 + \frac{c^2 - r^2}{y^2}\right) dy \, d\phi$$

$$= \frac{\pi \rho r dr}{c^2} \int_{c-r}^{c+r} \left(1 + \frac{c^2 - r^2}{y^2}\right) dy = \frac{\pi \rho r dr}{c^2} (2r + 2r)$$

$$= \frac{4\pi\rho r^2 dr}{c^2} = \frac{\text{mass of shell}}{c^2}.$$

This result shows that the shell attracts the particle at C in the same manner as if the mass of the shell were condensed into its centre.

3. It follows also that a sphere, which is either homogeneous or consists of concentric spherical shells of uniform density, will attract the particle C in the same manner as if the whole mass were collected at its centre.

PROP. *To find the attraction of a homogeneous spherical shell of small thickness on a particle situated within it.*

4. We must proceed as in the last Proposition; but the limits of y are in this case $r-c$ and $r+c$; hence,

$$\text{attraction of shell} = \frac{\pi\rho r dr}{c^2}\int_{r-c}^{r+c}\left(1-\frac{r^2-c^2}{y^2}\right)dy$$

$$= \frac{\pi\rho r dr}{c^2}(2c-2c) = 0;$$

therefore the particle within the shell is equally attracted in every direction.

5. This result may easily be arrived at geometrically in the following manner. Through the attracted point suppose an elementary double cone to be drawn, cutting the shell in two places. The inclinations of the elementary portions of the shell, thus cut out, to the axis of the cone will be the same, the thickness the same, but the other two dimensions of the elements will each vary as the distance from the attracted point; and therefore the masses of the two opposite elements of the shell will vary directly as the square of the distance from that point, and consequently their attractions will be exactly equal, and being in opposite directions will not affect the point. The whole shell may be thus divided into pairs of equal attracting elements and in opposite directions, and therefore the whole shell has no effect in drawing the point in any one direction more than in another.

6. The results of these two Propositions are so simple and beautiful, that it is interesting to enquire whether these

properties belong exclusively or not to the law of the inverse square of the distance. To determine this is the object of the four following Propositions.

PROP. *To find the attraction of a homogeneous spherical shell on a particle without it; the law of attraction being represented by* $\phi(y)$, *y being the distance.*

7. The calculation is exactly analogous to that given above: we have only to alter the law of attraction. Then attraction on C in CO

$$= \frac{\pi\rho r dr}{c^2} \int_{c-r}^{c+r} (y^2 + c^2 - r^2)\, \phi(y)\, dy \text{ (integrated by parts)}$$

$$= \frac{\pi\rho r dr}{c^2} \left[(y^2 + c^2 - r^2) \int \phi(y)\, dy - 2 \int \{y \int \phi(y)\, dy\}\, dy\right]$$

$$= \frac{\pi\rho r dr}{c^2} \{(y^2 + c^2 - r^2)\, \phi_1(y) - 2\psi(y) + \text{const.}\} \text{ suppose,}$$

$$= 2\pi\rho r dr \left\{\frac{c+r}{c} \phi_1(c+r) - \frac{1}{c^2} \psi(c+r) - \frac{c-r}{c} \phi_1(c-r) + \frac{1}{c^2} \psi(c-r)\right\}$$

$$= 2\pi\rho r dr \frac{d}{dc} \left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\},$$

this latter form being introduced merely as an analytical artifice to simplify the expression.

PROP. *To find the attraction of the shell on an internal particle, with the same law.*

8. The calculation is the same as in the last Article, except that the limits of y are $r-c$ and $r+c$:

$$\therefore \text{ attraction} = 2\pi\rho r dr \left\{\frac{r+c}{c} \phi_1(r+c) - \frac{1}{c^2} \psi(r+c) + \frac{r-c}{c} \phi_1(r-c) + \frac{1}{c^2} \psi(r-c)\right\}$$

$$= 2\pi\rho r dr \frac{d}{dc} \left\{\frac{\psi(r+c) - \psi(r-c)}{c}\right\}.$$

PROP. *To find what laws of attraction allow us to suppose a spherical shell condensed into its centre when attracting an external point.*

9. Let $\phi(r)$ be the law of force; then if c be the distance of the centre of the shell from the attracted point and r the radius of the shell, and

$$\psi(r) = \int \{r \int \phi(r)\, dr\}\, dr,$$

then the attraction of the shell

$$= 2\pi\rho r dr \frac{d}{dc}\left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\}.$$

But if the shell be condensed into its centre, the attraction

$$= 4\pi\rho r^2\, dr\, \phi(c);$$

$$\therefore\ 2r\phi(c) = \frac{d}{dc}\left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\}$$

$$= 2\frac{d}{dc}\left(\frac{d\psi c}{dc}\frac{r}{c} + \frac{d^3\psi c}{dc^3}\frac{r^3}{c}\frac{1}{1.2.3} + \ldots\right)$$

$$= 2r\phi(c) + 2\frac{d}{dc}\left(\frac{d^3\psi c}{dc^3}\frac{r^3}{c}\frac{1}{1.2.3} + \ldots\right);$$

$$\therefore\ \frac{d}{dc}\left(\frac{1}{c}\frac{d^3\psi c}{dc^3} + \ldots\right) = 0, \text{ whatever } r \text{ be};$$

$$\therefore\ \frac{d}{dc}\left(\frac{1}{c}\frac{d^3\psi c}{dc^3}\right) = 0, \quad \frac{d}{dc}\left(\frac{1}{c}\frac{d^5\psi c}{dc^5}\right) = 0, \ldots$$

But $\dfrac{d\psi c}{dc} = c\int\phi(c)\, dc$, $\dfrac{d^2\psi c}{dc^2} = \int\phi(c)\, dc + c\phi(c)$,

$$\frac{d^3\psi c}{dc^3} = 2\phi c + c\frac{d\phi c}{dc};$$

therefore by the first of the above equations of condition

$$\frac{2}{c}\phi c + \frac{d\phi c}{dc} = \text{const.} = 3A,$$

and multiplying by c^2 and integrating

$$c^2 \phi(c) = Ac^3 + B,$$

A and B being independent of c,

$$\phi(c) = Ac + \frac{B}{c^2}.$$

This is the most general solution of the first of the equations of condition for $\psi(c)$, and it satisfies all the rest. Hence the only laws of attraction which have the property in question are those of the direct distance, the inverse square, and a law compounded of these.

PROP. *To find for what laws the shell attracts an internal point equally in every direction.*

10. When this is the case

$$\frac{d}{dc}\left\{\frac{\psi(r+c) - \psi(r-c)}{c}\right\} = 0,$$

$$\frac{d\psi r}{dr} + \frac{d^3\psi r}{dr^3}\frac{c^2}{1.2.3} + \ldots = -A,$$

whatever c is, A being a constant independent of c;

$$\therefore \frac{d\psi r}{dr} = -A, \qquad \frac{d^3\psi r}{dr^3} = 0, \ldots$$

These conditions are all satisfied if the first is: this gives

$$r\int\phi(r)\,dr = -A, \qquad \phi(r) = \frac{A}{r^2},$$

and therefore the inverse square is the only law which possesses this property.

11. The form of the Earth and of the other bodies of the Solar System differing from the spherical, and more resembling the spheroidal, it is desirable to find the attraction of a spheroid upon an external and an internal point.

PROP. *To find the attraction of a homogeneous oblate spheroid upon a particle within its mass; the law of attraction being that of the inverse square of the distance.*

12. Let a, c be the semi-axes; the minor axis $2c$ coinciding with the axis of z: then the equation to the spheroid from the centre is

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1.$$

Let fgh be the co-ordinates to the attracted particle, which we shall take as the origin of polar co-ordinates,

r = radius vector of any particle of the attracting mass,

θ = angle which r makes with a line parallel to z,

ϕ = angle which the plane in which θ is measured makes with the plane xz;

$\therefore\ x=f+r\sin\theta\cos\phi,\ y=g+r\sin\theta\sin\phi,\ z=h+r\cos\theta,$

and the equation to the spheroid becomes

$$\frac{(f+r\sin\theta\cos\phi)^2+(g+r\sin\theta\sin\phi)^2}{a^2}+\frac{(h+r\cos\theta)^2}{c^2}=1,$$

$$\text{or } r^2\left(\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{c^2}\right)+2r\left(\frac{f\sin\theta\cos\phi+g\sin\theta\sin\phi}{a^2}+\frac{h\cos\theta}{c^2}\right)$$

$$=1-\frac{f^2+g^2}{a^2}-\frac{h^2}{c^2};$$

$$\text{put}\quad \frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{c^2}=K,$$

$$\frac{f\sin\theta\cos\phi+g\sin\theta\sin\phi}{a^2}+\frac{h\cos\theta}{c^2}=F,$$

$$\text{and}\quad F^2+K\left(1-\frac{f^2+g^2}{a^2}-\frac{h^2}{c^2}\right)=H,$$

$$\text{then}\quad K^2r^2+2KFr+F^2=H,$$

and the values of r are

$$r'=\frac{-F+\sqrt{(H)}}{K}\quad\text{and}\quad r''=\frac{-F-\sqrt{(H)}}{K}.$$

The volume of the attracting element $= r^2 \sin\theta\, dr\, d\theta\, d\phi$ as in Art. 2: let ρ be the density of the spheroid. Then the attraction of this element on the attracted particle is

$$\rho \sin\theta\, dr\, d\theta\, d\phi :$$

and the resolved parts of this parallel to the axes of xyz are

$$\rho \sin^2\theta \cos\phi\, dr\, d\theta\, d\phi, \quad \rho \sin^2\theta \sin\phi\, dr\, d\theta\, d\phi,$$

$$\rho \sin\theta \cos\theta\, dr\, d\theta\, d\phi.$$

Let A, B, C be the attractions of the whole spheroid in the directions of the axes, estimated positive towards the centre of the spheroid. Then these equal the integrals of the attractions of the element; the limits of r being $-r'$ and r'', of θ being 0 and π, of ϕ being 0 and π. Hence

$$A = -\int_{-r'}^{r''}\int_0^\pi\int_0^\pi \rho \sin^2\theta \cos\phi\, dr\, d\theta\, d\phi,$$

$$B = -\int_{-r'}^{r''}\int_0^\pi\int_0^\pi \rho \sin^2\theta \sin\phi\, dr\, d\theta\, d\phi,$$

$$C = -\int_{-r'}^{r''}\int_0^\pi\int_0^\pi \rho \sin\theta \cos\theta\, dr\, d\theta\, d\phi.$$

Then
$$A = -\rho\int_0^\pi\int_0^\pi (r'' + r') \sin^2\theta \cos\phi\, d\theta\, d\phi$$

$$= 2\rho\int_0^\pi\int_0^\pi \frac{F}{K} \sin^2\theta \cos\phi\, d\theta\, d\phi.$$

Now it is easily seen that if $R(\sin\alpha, \cos^2\alpha)$ be a rational function of $\sin\alpha$ and $\cos^2\alpha$, then

$$\int_0^\pi R(\sin\alpha, \cos^2\alpha) \cos\alpha\, d\alpha = 0.$$

Therefore by substituting for F and K we have

$$A = 2f\rho c^2 \int_0^\pi\int_0^\pi \frac{\sin^3\theta \cos^2\phi\, d\theta\, d\phi}{c^2 \sin^2\theta + a^2 \cos^2\theta}$$

$$= \pi f\rho c^2 \int_0^\pi \frac{\sin^3\theta d\theta}{c^2\sin^2\theta + a^2\cos^2\theta} = \pi f\rho c^2 \int_0^\pi \frac{(1-\cos^2\theta)\sin\theta d\theta}{c^2 + (a^2-c^2)\cos^2\theta} *$$

$$= \pi f\rho \frac{c^2}{a^2-c^2}\int_0^\pi \left\{\frac{a^2\sin\theta}{c^2+(a^2-c^2)\cos^2\theta} - \sin\theta\right\} d\theta$$

$$= \pi f\rho \frac{c^2}{a^2-c^2}\left\{-\frac{a^2}{c\sqrt{a^2-c^2}}\tan^{-1}\left(\frac{\sqrt{a^2-c^2}}{c}\cos\theta\right) + \cos\theta + \text{const.}\right\}$$

$$= 2\pi f\rho \frac{c^2}{a^2-c^2}\left\{\frac{a^2}{c\sqrt{a^2-c^2}}\tan^{-1}\frac{\sqrt{a^2-c^2}}{c} - 1\right\}, \quad \frac{c^2}{a^2} = 1-e^2,$$

$$= 2\pi f\rho \left\{\frac{\sqrt{1-e^2}}{e^3}\tan^{-1}\frac{e}{\sqrt{1-e^2}} - \frac{1-e^2}{e^2}\right\}$$

$$= 2\pi f\rho \left\{\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e - \frac{1-e^2}{e^2}\right\}.$$

In the same manner we should find that

$$B = 2\pi g\rho \left\{\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e - \frac{1-e^2}{e^2}\right\}.$$

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Also $$C = 2\rho\int_0^\pi\int_0^\pi \frac{F}{K}\sin\theta\cos\theta\, d\theta\, d\phi$$

$$= 2\rho h a^2 \int_0^\pi\int_0^\pi \frac{\sin\theta\cos^2\theta\, d\theta\, d\phi}{c^2\sin^2\theta + a^2\cos^2\theta}$$

$$= 2\pi\rho h \frac{a^2}{a^2-c^2}\int_0^\pi \left\{\sin\theta - \frac{c^2\sin\theta}{c^2+(a^2-c^2)\cos^2\theta}\right\} d\theta$$

$$= 4\pi\rho h \frac{a^2}{a^2-c^2}\left\{1 - \frac{c}{\sqrt{a^2-c^2}}\tan^{-1}\frac{\sqrt{a^2-c^2}}{c}\right\}$$

$$= 4\pi\rho h \left\{\frac{1}{e^2} - \frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e\right\}.$$

* If the spheroid be prolate, c is $> a$ and the denominator of this must be written $c^2 - (c^2 - a^2)\cos^2\theta$, and the integral would involve logarithms instead of circular arcs.

13. Cor. 1. We gather from these expressions, that the attraction is independent of the magnitude of the spheroid, and depends solely upon its ellipticity. Hence the attraction of the spheroid similar to the given one, and passing through the attracted particle, is the same as that of any other similar concentric spheroid comprising the attracted particle in its mass. Hence a spheroidal shell, the surfaces of which are similar and concentric, attracts a point within it equally in all directions.

14. Cor. 2. If we put the ellipticity of the spheroid $= \epsilon$, and suppose ϵ so small that we may neglect its square, we have

$$e^2 = 1 - \frac{c^2}{a^2} = 1 - (1-\epsilon)^2 = 2\epsilon;$$

$$\therefore A = \frac{4}{3}\pi\rho\left(1 - \frac{2}{5}\epsilon\right)f, \quad B = \frac{4}{3}\pi\rho\left(1 - \frac{2}{5}\epsilon\right)g,$$

$$C = \frac{4}{3}\pi\rho\left(1 + \frac{4}{5}\epsilon\right)h.$$

If we had taken an ellipsoid instead of a spheroid, the expressions would not have been capable of integration.

15. If we had attempted to find the attraction on an external particle according to the process of the last Article, we should have fallen upon expressions which no known methods have yet integrated: and therefore we are unable by any direct means to obtain the attraction of a spheroid on an external particle. Mr Ivory has, however, devised an indirect method of obtaining it, which we shall now proceed to develop. He has discovered a theorem by which the attraction of an ellipsoid upon an external particle is shown to be proportional to that of another ellipsoid, dependent on the first for form and dimensions, upon a particle internal to it, and therefore (in the case of a spheroid, or ellipsoid of revolution) determinable by the last Proposition.

Prop. *To enunciate and prove Ivory's Theorem.*

16. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$,

be the equations to the surfaces of two ellipsoids having the same centre and foci: then

$$a^2 - b^2 = \alpha^2 - \beta^2, \quad a^2 - c^2 = \alpha^2 - \gamma^2 \ldots\ldots\ldots\ldots (1).$$

Let fgh, $f'g'h'$ be the co-ordinates to two particles so situated on the surfaces of these ellipsoids that

$$\frac{f}{f'} = \frac{a}{\alpha}, \quad \frac{g}{g'} = \frac{b}{\beta}, \quad \frac{h}{h'} = \frac{c}{\gamma} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also since (fgh) $(f'g'h')$ are points in the surfaces of the first and second ellipsoids respectively, we have

$$\frac{f^2}{a^2} + \frac{g^2}{b^2} + \frac{h^2}{c^2} = 1, \quad \frac{f'^2}{\alpha^2} + \frac{g'^2}{\beta^2} + \frac{h'^2}{\gamma^2} = 1 \ldots\ldots\ldots\ldots (3).$$

Then the attraction of the first ellipsoid parallel to the axis of x on the particle at the point $(f'g'h')$ on the surface of the second, is to the attraction of the second ellipsoid on the particle at the point (fgh) on the surface of the first in the same direction, as $ab : \alpha\beta$, the law of attraction being any function of the distance: and similarly with respect to the axes of y and z. This is Ivory's Theorem.

We shall, for convenience, represent the law of attraction by the function $r\phi(r^2)$, r being the distance.

The attraction of the first ellipsoid on the particle $(f'g'h')$ parallel to the axis of z

$$= \rho \iiint (h' - z)\, \phi \{(f' - x)^2 + (g' - y)^2 + (h' - z)^2\}\, dx\, dy\, dz,$$

the limits of z are $-c\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}$, and $c\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}$,

the limits of y are $-b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}$, and $b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}$,

and the limits of x are $-a$ and a

$$= \rho \iint [\psi \{(f' - x)^2 + (g' - y)^2 + (h' + z)^2\} - \psi \{(f' - x)^2 + (g' - y)^2 + (h' - z)^2\}]\, dx\, dy$$

between the specified limits:

$$\psi(r) = \frac{1}{2}\int \phi(r)\, dr:$$

it must be remembered that in this expression

$$z = c\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)},$$

but we do not substitute this value merely that the function may be preserved under as simple a form as possible. Now put $x = ar$, $y = bs$, $z = ct$, then the attraction

$$= \rho ab \iint [\psi\{(f' - ar)^2 + (g' - bs)^2 + (h' + ct)^2\} - \psi\{(f' - ar)^2 + (g' - bs)^2 + (h' - ct)^2\}]\, dr\, ds,$$

the limits of s being $-\sqrt{(1-r^2)}$ and $\sqrt{(1-r^2)}$, and those of r being -1 and 1: also $t = \sqrt{(1-r^2-s^2)}$.

Now $(f' - ar)^2 + (g' - bs)^2 + (h' \pm ct)^2$

$$= f'^2 + g'^2 + h'^2 - 2(f'ar + g'bs \pm h'ct) + a^2r^2 + b^2s^2 + c^2t^2,$$

substituting for h'^2 by (3) and for t^2,

$$= f'^2\left(1 - \frac{\gamma^2}{\alpha^2}\right) + g'^2\left(1 - \frac{\gamma^2}{\beta^2}\right) + \gamma^2 - 2(f'ar + g'bs \pm h'ct) + (a^2 - c^2)\, r^2 + (b^2 - c^2)\, s^2 + c^2,$$

eliminating $f'g'h'$ by (2) and making use of (1),

$$= \frac{f^2}{a^2}(a^2 - c^2) + \frac{g^2}{b^2}(b^2 - c^2) + c^2 - 2(f\alpha r + g\beta s \pm h\gamma t) + (\alpha^2 - \gamma^2)\, r^2 + (\beta^2 - \gamma^2)\, s^2 + \gamma^2$$

$$= f^2 + g^2 + h^2 - 2(f\alpha r + g\beta s \pm h\gamma t) + \alpha^2r^2 + \beta^2s^2 + \gamma^2t^2, \text{ by (3)},$$

$$= (f - \alpha r)^2 + (g - \beta s)^2 + (h \pm \gamma t).$$

Hence the attraction of the First Ellipsoid on $(f'g'h')$ parallel to z,

$$= \rho ab \iint [\psi\{(f - \alpha r)^2 + (g - \beta s)^2 + (h + \gamma t)^2\} - \psi\{(f - \alpha r)^2 + (g - \beta s)^2 + (h - \gamma t)^2\}]\, dr\, ds$$

$= \frac{ab}{a\beta} \times$ attraction of Second Ellipsoid on (fgh) in the same direction.

The same may be proved for the attractions parallel to the other axes: and consequently the Theorem, as enunciated, is true.

We may observe that one of these ellipsoids must necessarily be wholly within the other. For if not, the points in which they cut each other lie in the line of which the equations are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{and} \quad \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1.$$

Suppose a less than α; the points of intersection must satisfy the equation

$$x^2\left(\frac{1}{a^2}-\frac{1}{\alpha^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{\beta^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{\gamma^2}\right)=0;$$

and this by (1) becomes

$$\left(\frac{x}{a\alpha}\right)^2+\left(\frac{y}{b\beta}\right)^2+\left(\frac{z}{c\gamma}\right)^2=0,$$

an equation which can be satisfied only by $x=0$, $y=0$, $z=0$. But these do not satisfy the equations above; and therefore the surfaces do not intersect in any point.

To find the attraction of any ellipsoid of which the semi-axes are a, b, c upon an external point $(f'g'h')$ by the help of this Theorem, we must first calculate the attraction of an ellipsoid of which the semi-axes are $\alpha\beta\gamma$, determined by equations (1) and the second of (3), on an internal point (fgh), f, g and h being given by equations (2). And then the attractions required will be those multiplied by

$$\frac{bc}{\beta\gamma}, \quad \frac{ac}{\alpha\gamma}, \quad \frac{ab}{\alpha\beta}, \text{ respectively.}$$

CHAPTER II.

LAPLACE'S COEFFICIENTS AND FUNCTIONS.

17. In the present Chapter we shall develop the properties of those remarkable quantities which have received the name of their great discoverer, under the designation of LAPLACE'S COEFFICIENTS AND FUNCTIONS. To do this it will be necessary to anticipate the subject of the following Chapter, and to bring in here a Proposition which should properly stand at the head of that division of this treatise.

PROP. *To obtain formulæ for the calculation of the attraction of a heterogeneous mass upon any particle.*

18. Let ρ be the density of the body at the point (xyz); fgh the co-ordinates of the attracted particle; and, as before, suppose that A, B, C are the attractions parallel to the axes x, y, z. Then

$$A = \iiint \frac{\rho\,(f-x)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{3}{2}}},$$

$$B = \iiint \frac{\rho\,(g-y)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{3}{2}}},$$

$$C = \iiint \frac{\rho\,(h-z)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+h-z)^2\}^{\frac{3}{2}}},$$

the limits being determined by the equation to the surface of the body.

Let $V = \iiint \frac{\rho\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{1}{2}}}$;

$$\therefore\ A = -\frac{dV}{df},\quad B = -\frac{dV}{dg},\quad C = -\frac{dV}{dh}.$$

19. It follows, then, that the calculation of the attractions A, B, C depends upon that of V. This function cannot be

calculated except when expanded into a series. It is a function of great importance in Physics: and, for the sake of a name, has been denominated the *Potential* of the attracting mass, as upon its value the amount of the attractive force of the body depends.

20. As the axes and origin of co-ordinates in the previous Article are altogether arbitrary, it follows that if r be the distance of the attracted point from any fixed point in the attracting body, then the attraction in the line of r, towards the origin of r, $= -\dfrac{dV}{dr}$.

PROP. *To prove that* $\dfrac{d^2V}{df^2} + \dfrac{d^2V}{dg^2} + \dfrac{d^2V}{dh^2} = 0$, *or* $-4\pi\rho'$, *according as the attracted particle is not or is part of the mass itself;* ρ' *being the density of the attracted particle in the latter case.*

21. By differentiating V, we have

$$\frac{dV}{df} = \iiint \frac{-\rho\,(f-x)\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{3}{2}}},$$

$$\frac{d^2V}{df^2} = \iiint \frac{\rho\,\{2\,(f-x)^2-(g-y)^2-(h-z)^2\}\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{5}{2}}}.$$

In the same manner we shall have

$$\frac{d^2V}{dg^2} = \iiint \frac{\rho\,\{2\,(g-y)^2-(f-x)^2-(h-z)^2\}\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{5}{2}}},$$

$$\frac{d^2V}{dh^2} = \iiint \frac{\rho\,\{2\,(h-z)^2-(f-x)^2-(g-y)^2\}\,dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{5}{2}}};$$

$$\therefore \frac{d^2V}{df^2} + \frac{d^2V}{dg^2} + \frac{d^2V}{dh^2} = \iiint \frac{0 \times dx\,dy\,dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{5}{2}}}.$$

When the attracted particle is not a portion of the attracting mass itself, then xyz will never equal fgh respectively,

and consequently the expression under the signs of integration vanishes for every particle of the mass:

$$\therefore \frac{d^2V}{df^2}+\frac{d^2V}{dg^2}+\frac{d^2V}{dh^2}=0.$$

This equation was first given by Laplace: and Poisson was the first who showed that it is not true when the attracted particle is part of the attracting mass. In that case the denominator of the fraction under the signs of integration vanishes, and the fraction becomes $\frac{0}{0}$, when $x=f,\ y=g,\ z=h$.

To determine the value of $\frac{d^2V}{df^2}+\frac{d^2V}{dg^2}+\frac{d^2V}{dh^2}$ in that case, suppose a sphere described in the body, so that it shall include the attracted particle; and let $V=U+U'$, U referring to the sphere, and U' to the excess of the body over the sphere. Then, by what is already proved,

$$\frac{d^2U'}{df^2}+\frac{d^2U'}{dg^2}+\frac{d^2U'}{dh^2}=0;$$

$$\therefore \frac{d^2V}{df^2}+\frac{d^2V}{dg^2}+\frac{d^2V}{dh^2}=\frac{d^2U}{df^2}+\frac{d^2U}{dg^2}+\frac{d^2U}{dh^2}.$$

The centre of the sphere may be chosen as near the attracted particle as we please; and therefore the radius of the sphere may be taken so small that its density may be considered uniform and equal to that at the point (fgh), which we shall call ρ'.

Let $f'g'h'$ be the co-ordinates to the centre of the sphere; then the attractions of the sphere on the attracted point parallel to the axes are, by Art. 3,

$$\frac{4\pi\rho'}{3}(f-f'),\ \frac{4\pi\rho'}{3}(g-g'),\ \frac{4\pi\rho'}{3}(h-h'),$$

$$\text{or } -\frac{dU}{df},\ -\frac{dU}{dg},\ -\frac{dU}{dh},\ \text{by Art. 20.}$$

$$\therefore \frac{d^2U}{df^2}+\frac{d^2U}{dg^2}+\frac{d^2U}{dh^2}=-4\pi\rho';$$

$$\therefore \frac{d^2V}{df^2}+\frac{d^2V}{dg^2}+\frac{d^2V}{dh^2}=-4\pi\rho',$$

when the attracted particle is within the attracting mass.

22. It may be shown by precisely the same process as in the previous Article, that

$$\frac{d^2R}{df^2}+\frac{d^2R}{dg^2}+\frac{d^2R}{dh^2}=0,$$

where $R=\{(f-x)^2+(g-y)^2+(h-z)^2\}^{-\frac{1}{2}}$,

the reciprocal of the distance of any point of the body from the attracted particle.

PROP. *To transform the partial differential equation in R into polar co-ordinates.*

23. Let $r\theta\omega$ be the co-ordinates of (fgh), and $r'\theta'\omega'$ of (xyz), the angles θ and θ' being measured from the axis of z; ω and ω' being the angles which the planes on which θ and θ' are measured make with the plane zx. Then

$$f=r\sin\theta\cos\omega,\quad g=r\sin\theta\sin\omega,\quad h=r\cos\theta,$$

$$x=r'\sin\theta'\cos\omega',\quad g'=r'\sin\theta'\sin\omega',\quad h'=r'\cos\theta'.$$

These are the same as

$$r^2=f^2+g^2+h^2,\quad \cos\theta=\frac{h}{\sqrt{f^2+g^2+h^2}},\quad \tan\omega=\frac{g}{f}\ldots\ldots(1);$$

$$\therefore \frac{dR}{df}=\frac{dR}{dr}\frac{dr}{df}+\frac{dR}{d\theta}\frac{d\theta}{df}+\frac{dR}{d\omega}\frac{d\omega}{df},$$

$$\frac{d^2R}{df^2}=\frac{d}{df}\frac{dR}{dr}\frac{dr}{df}+\frac{d}{df}\frac{dR}{d\theta}\frac{d\theta}{df}+\frac{d}{df}\frac{dR}{d\omega}\frac{d\omega}{df}$$

$$+\frac{dR}{dr}\frac{d^2r}{df^2}+\frac{dR}{d\theta}\frac{d^2\theta}{df^2}+\frac{dR}{d\omega}\frac{d^2\omega}{df^2}$$

$$= \frac{d^2R}{dr^2}\frac{dr^2}{df^2} + \frac{d^2R}{d\theta^2}\frac{d\theta^2}{df^2} + \frac{d^2R}{d\omega^2}\frac{d\omega^2}{df^2}$$

$$+ 2\frac{d^2R}{dr\,d\theta}\frac{dr}{df}\frac{d\theta}{df} + 2\frac{d^2R}{dr\,d\omega}\frac{dr}{df}\frac{d\omega}{df} + 2\frac{d^2R}{d\theta\,d\omega}\frac{d\theta}{df}\frac{d\omega}{df}$$

$$+ \frac{dR}{dr}\frac{d^2r}{df^2} + \frac{dR}{d\theta}\frac{d^2\theta}{df^2} + \frac{dR}{d\omega}\frac{d^2\omega}{df^2}.$$

The expressions for $\frac{d^2R}{dg^2}$ and $\frac{d^2R}{dh^2}$ are of the same form. These three must be added together and equated to zero. When this is effected the formulæ (1) make

the coefficient of $\frac{d^2R}{dr^2} = \frac{dr^2}{df^2} + \frac{dr^2}{dg^2} + \frac{dr^2}{dh^2} = 1,$

the coefficient of $\frac{d^2R}{d\theta^2} = \frac{d\theta^2}{df^2} + \frac{d\theta^2}{dg^2} + \frac{d\theta^2}{dh^2} = \frac{1}{r^2},$

the coefficient of $\frac{d^2R}{d\omega^2} = \frac{d\omega^2}{df^2} + \frac{d\omega^2}{dg^2} + \frac{d\omega^2}{dh^2} = \frac{1}{r^2\sin^2\theta},$

the coefficient of $\frac{d^2R}{dr\,d\theta} = 2\frac{dr}{df}\frac{d\theta}{df} + 2\frac{dr}{dg}\frac{d\theta}{dg} + 2\frac{dr}{dh}\frac{d\theta}{dh} = 0,$

the coefficient of $\frac{d^2R}{dr\,d\omega} = 2\frac{dr}{df}\frac{d\omega}{df} + 2\frac{dr}{dg}\frac{d\omega}{dg} + 2\frac{dr}{dh}\frac{d\omega}{dh} = 0,$

the coefficient of $\frac{d^2R}{d\theta\,d\omega} = 2\frac{d\theta}{df}\frac{d\omega}{df} + 2\frac{d\theta}{dg}\frac{d\omega}{dg} + 2\frac{d\theta}{dh}\frac{d\omega}{dh} = 0,$

the coefficient of $\frac{dR}{dr} = \frac{d^2r}{df^2} + \frac{d^2r}{dg^2} + \frac{d^2r}{dh^2} = \frac{2}{r},$

the coefficient of $\frac{dR}{d\theta} = \frac{d^2\theta}{df^2} + \frac{d^2\theta}{dg^2} + \frac{d^2\theta}{dh^2} = \frac{\cos\theta}{r^2\sin\theta},$

the coefficient of $\frac{dR}{d\omega} = \frac{d^2\omega}{df^2} + \frac{d^2\omega}{dg^2} + \frac{d^2\omega}{dh^2} = 0.$

Hence the equation in R becomes

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\frac{1}{r^2}\frac{d^2R}{d\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{dR}{d\theta}+\frac{1}{r^2\sin^2\theta}\frac{d^2R}{d\omega^2}=0;$$

$$\therefore\ r\frac{d^2.rR}{dr^2}+\frac{d^2R}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{dR}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2R}{d\omega^2}=0.$$

Put $\cos\theta=\mu$, then

$$r\frac{d^2.rR}{dr^2}+\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR}{d\mu}\right\}+\frac{1}{1-\mu^2}\frac{d^2R}{d\omega^2}=0.$$

24. Cor. The same equation is true for V, for an external particle. If the particle be part of the attracting mass, the second side will be $-4\pi\rho'$ instead of zero.

Prop. *To explain the method of expanding R in a series.*

25. The expression for R becomes, when the polar co-ordinates are substituted,

$$[r^2+r'^2-2rr'\{\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega')\}]^{-\frac{1}{2}},$$

and this may be expanded into either of the series

$$\left.\begin{aligned} &P_0\frac{1}{r'}+P_1\frac{r}{r'^2}+\ldots\ldots\ldots+P_i\frac{r^i}{r'^{i+1}}+\ldots\\ \text{or }\ &P_0\frac{1}{r}+P_1\frac{r'}{r^2}+\ldots\ldots\ldots+P_i\frac{r'^i}{r^{i+1}}+\ldots\end{aligned}\right\}\ \ldots\ldots(1),$$

where P_0, P_1, ...P_i... are all determinate rational and entire functions of μ,

$$\sqrt{1-\mu^2}\cos\omega,\ \text{and}\ \sqrt{1-\mu^2}\sin\omega;$$

and the same functions of μ',

$$\sqrt{1-\mu'^2}\cos\omega',\ \text{and}\ \sqrt{1-\mu'^2}\sin\omega'.$$

The general coefficient P_i is of i dimensions in μ,

$$\sqrt{1-\mu^2}\cos\omega,\ \text{and}\ \sqrt{1-\mu^2}\sin\omega.$$

The greatest value of P_i (disregarding its sign) is unity. For if we put

$$\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega') = \cos\phi = \tfrac{1}{2}\left(z + \frac{1}{z}\right),$$

then P_i = coefficient of c^i in

$$(1 + c^2 - 2c\cos\phi)^{-\frac{1}{2}}, \text{ or } (1-cz)^{-\frac{1}{2}}\left(1 - \frac{c}{z}\right)^{-\frac{1}{2}}$$

= coefficient of c^i in

$$\left(1 + \frac{1}{2}cz + \frac{1.3}{2.4}c^2z^2 + \ldots\right)\left(1 + \frac{1}{2}\frac{c}{z} + \frac{1.3}{2.4}\frac{c^3}{z^2} + \ldots\right)$$

$$= A\left(z^i + \frac{1}{z^i}\right) + B\left(z^{i-2} + \frac{1}{z^{i-2}}\right) + \ldots$$

$$= 2A\cos i\phi + 2B\cos(i-2)\phi + \ldots$$

A, B... being all positive and finite. The greatest value of this is, when $\phi = 0$. Hence P_i is greatest when $\phi = 0$.

But then P_i = coefficient of c^i in $(1 + c^2 - 2c)^{-\frac{1}{2}}$ or $(1-c)^{-1}$

$$= \text{coefficient of } c^i \text{ in } 1 + c + c^2 + \ldots + c^i + \ldots$$

$$= 1.$$

Hence 1 is the greatest value of P_i. It follows that the first or second of series (1) will be convergent according as r is less than or greater r'.

To obtain equations for calculating the coefficients $P_0, P_1, \ldots$ $P_i \ldots$ substitute either of the series (1) in the differential equation in R in the last article, and equate the coefficients of the several powers of r to zero. The general term gives the following equation:

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_i}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2P_i}{d\omega^2} + i(i+1)P = 0,$$

by integrating which P_i should be determined*. The series for R would then be known.

* For the direct integration of this equation, see two Papers in the *Philosophical Transactions* for 1841 and 1857, by Mr Hargreave and Professor Donkin respectively.

26. The functions P_0, $P_1 \ldots P_i \ldots$ possess some remarkable properties which were discovered by Laplace. They are therefore called, after him, *Laplace's Coefficients*, of the orders 0, 1,... i... It will be observed that these quantities are definite and have no arbitrary constants in them. Laplace's Coefficients are therefore certain definite expressions involving only numerical quantities with μ and ω, μ' and ω'. Any other expressions which may satisfy the partial differential equation in P_i, which is called Laplace's Equation, may be designated *Laplace's Functions* to distinguish them from the "Coefficients." The fundamental properties of these Coefficients and Functions we shall now proceed to demonstrate.

PROP. *To prove that if Q_i and $R_{i'}$ be two Laplace's Coefficients or Functions, then* $\int_{-1}^{1}\int_{0}^{2\pi} Q_i R_{i'} d\mu\, d\omega = 0$, *when i and i' are different integers.*

27. By Laplace's Equation in the last Article

$$i(i+1)\, Q_i = -\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} - \frac{1}{1-\mu^2}\frac{d^2 Q_i}{d\omega^2};$$

$$\therefore \int_{-1}^{1}\int_{0}^{2\pi} Q_i R_{i'} d\mu\, d\omega$$

$$= -\frac{1}{i(i+1)}\int_{-1}^{1}\int_{0}^{2\pi}\left[\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2 Q_i}{d\omega^2}\right] R_{i'} d\mu\, d\omega.$$

By a double integration by parts

$$\int \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} R_{i'} d\mu = (1-\mu^2)\frac{dQ_i}{d\mu} R_{i'} - (1-\mu^2)\frac{dR_{i'}}{d\mu} Q_i$$

$$+ \int \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR_{i'}}{d\mu}\right\} Q_i d\mu;$$

$$\therefore \int_{-1}^{1} \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} R_{i'} d\mu = \int_{-1}^{1} \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR_{i'}}{d\mu}\right\} Q_i d\mu.$$

Again, $\int R_{i'}\frac{d^2 Q_i}{d\omega^2} d\omega = R_{i'}\frac{dQ_i}{d\omega} - Q_i\frac{dR_{i'}}{d\omega} + \int Q\frac{d^2 R_{i'}}{d\omega^2} d\omega;$

$$\therefore \int_0^{2\pi} R_{i'} \frac{d^2 Q_i}{d\omega^2}\, d\omega = \int_0^{2\pi} Q_i \frac{d^2 R_{i'}}{d\omega^2}\, d\omega,$$

since when $\omega = 0$ and 2π, each of the functions Q_i, $R_{i'}$, $\frac{dQ_i}{d\omega}$, $\frac{dR_{i'}}{d\omega}$ has the same values, because they are functions of μ,

$$\sqrt{1-\mu^2}\cos\omega \text{ and } \sqrt{1-\mu^2}\sin\omega.$$

Hence, $\int_{-1}^{1}\int_0^{2\pi} Q_i R_{i'}\, d\mu\, d\omega$

$$= -\frac{1}{i(i+1)}\int_{-1}^{1}\int_0^{2\pi}\left[\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR_{i'}}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2 R_{i'}}{d\omega^2}\right] Q_i\, d\mu\, d\omega$$

$$= \frac{i'(i'+1)}{i(i+1)}\int_{-1}^{1}\int_0^{2\pi} Q_i R_{i'}\, d\mu\, d\omega,$$

by Laplace's Equation.

Hence, $\int_{-1}^{1}\int_0^{2\pi} Q_i R_{i'}\, d\mu\, d\omega = 0$, when i and i' are unequal. When they are the same the equation becomes an identical one, and therefore gives no result.

This property is true also when $i = 0$, as may easily be shown by going through the process of the last Proposition, Q_i being Q_0 or a constant.

PROP. *To prove that a function of* μ, $\sqrt{1-\mu^2}\cos\omega$, *and* $\sqrt{1-\mu^2}\sin\omega$, *as* $F(\mu, \omega)$, *can be expanded in a series of Laplace's Functions; provided that* $F(\mu, \omega)$ *do not become infinite between the limits* -1 *and* 1 *of* μ, *and* 0 *and* 2π *of* ω.

28. This very important Proposition will occupy the present and four following Articles.

Let $\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega') = p$: then by Art. 25,

$$(1 + c^2 - 2cp)^{-\frac{1}{2}} = 1 + P_1 c + P_2 c^2 + \ldots\ldots P_i c^i + \ldots\ldots$$

c being any quantity not greater than unity.

Differentiate with respect to c,

$$\frac{p-c}{(1+c^2-2cp)^{\frac{3}{2}}} = P_1 + 2P_2c + \ldots\ldots + iP_ic^{i-1} + \ldots\ldots$$

Multiply this by $2c$ and add to it the former equation;

$$\therefore \frac{1-c^2}{(1+c^2-2cp)^{\frac{3}{2}}} = 1 + 3P_1c + 5P_2c^2 + \ldots\ldots + (2i+1)\,P_ic_i + \ldots$$

Now c being quite arbitrary we may put it $=1$. Then the fraction on the left-hand side of this equation vanishes, except when $p=1$; in which case the fraction on the left hand becomes apparently indeterminate: but it is in reality infinite.

For when $p=1$, $\dfrac{1-c^2}{(1+c^2-2cp)^{\frac{3}{2}}} = \dfrac{1+c}{(1-c)^2} =$ infinity, when $c=1$.

When $p=1$, then $\mu' = \mu$ and $\omega' = \omega$. For when $p=1$

$$\cos(\omega'-\omega) = \frac{1-\mu\mu'}{\sqrt{(1-\mu^2)\,(1-\mu'^2)}} = \sqrt{\frac{1-2\mu\mu'+\mu^2\mu'^2}{1-\mu^2-\mu'^2+\mu^2\mu'^2}},$$

and that this may not be greater than unity we must take $\mu^2+\mu'^2$ not greater than $2\mu\mu'$, or $(\mu\sim\mu')^2$ not greater than zero. Hence $\mu'=\mu$, and therefore $\cos(\omega'-\omega)=1$, and $\omega'=\omega$.

Hence, the series $1+3P_1+5P_2+\ldots\ldots+(2i+1)\,P_i+\ldots\ldots$ vanishes for all values of μ and ω, μ' and ω', except when $\mu=\mu'$ and $\omega=\omega'$, in which case the sum of its terms suddenly changes from zero to infinity.

29. Upon this series depends the important property of Laplace's Functions which we are now demonstrating, and which gives them so great a value in the higher branches of analysis. Our demonstration consists in showing, that

$$\int_{-1}^{1}\int_{0}^{2\pi} \frac{(1-c^2)\,d\mu'\,d\omega'\,F(\mu',\omega')}{(1+c^2-2cp)^{\frac{3}{2}}} = 4\pi F(\mu,\omega) \text{ when } c=1,$$

and that consequently

$$4\pi F(\mu,\omega) = \int_{-1}^{1}\int_{0}^{2\pi} (1+3P_1+5P_2+\ldots)\,F(\mu',\omega')\,d\mu'\,d\omega';$$

from which property, as will be seen in the end, our Proposition, as enunciated, immediately flows.

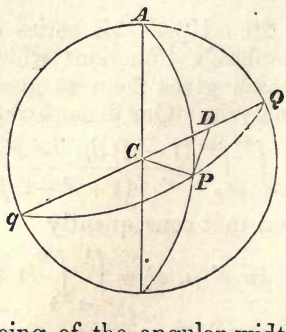
In consequence of the discontinuity above pointed out in the series $1 + 3P_1 + \ldots$, and also because the series becomes infinite in one stage of the variation of its variables, it has been considered by some to be unsatisfactory to deduce any properties from it. But the latter objection is entirely removed by the fact, that we do not use the series in its present form, but after being multiplied by the small infinitesimal quantities $d\mu'$, $d\omega'$ which, as will be seen in the next Art., makes the aggregate of its terms finite, preventing their accumulating to an infinite amount. With regard to the objection of discontinuity, there appears to be no sufficient ground for it. There is no question, that the property deduced (as enunciated in our Proposition) is true, at any rate for rational functions of μ, $\sqrt{1-\mu^2}\cos\omega$, and $\sqrt{1-\mu^2}\sin\omega$, and is also most important. This objection, however, deserves to be examined with care, which we now propose to do in the course of our demonstration.

30. We shall first prove, that

$$\int_{-1}^{1}\int_{0}^{2\pi} \frac{(1-c^2)\,d\mu' d\omega'}{(1+c^2-2cp)^{\frac{3}{2}}} = 4\pi.$$

This integration cannot be effected with the co-ordinates as at present chosen. But it may be done by a simple transformation, and a change in the way of taking the elements.

Suppose a sphere of radius unity described about C the origin of co-ordinates. Let θ' and ω' be the angular co-ordinates to a point P, θ' (or $\cos^{-1}\mu'$) measured from a fixed point A along a great circle of the sphere, and ω' the angle which this great circle makes with another and fixed great circle through A. Then $d\theta' . d\omega' \sin\theta'$, or $-d\mu' d\omega'$, is an infinitesimal element of the surface of the sphere at P. This division of the surface into elements supposes it to be cut into lunes from A to a, each being of the angular width $d\omega'$; and these into elements by parallel planes perpendicular



to ACa, at a distance $d\mu'$ from each other. The elements thus formed, though of different shapes, are all equal to each other in area. Take D a point within the sphere, and let $CD = c$, and suppose CD meets the sphere in Q when produced forwards, and in q when produced backwards. Let μ and ω be the co-ordinates of Q. Then p (see its value, Art. 28) is the cosine of the angle which CP and CQ make with each other: and the distance of P from $D = \sqrt{1 + c^2 - 2cp}$. Let ψ be the angle which the plane CPQ makes with CAQ, that is, the angle AQP. By changing the origin of the angles from A to Q, and dividing the surface of the sphere into new elements, beginning from Q as the origin, the element at P, with these new co-ordinates $\cos^{-1} p$ and ψ, will be $-dpd\psi$, and will $= -d\mu' d\omega'$, by choosing these increments right.

By reverting to the meaning of integration we see that the integral under consideration $= -(1 - c^2) \times$ limit of sum of all the elements of the surface of the sphere divided respectively by the cubes of their distances from D.

But this, by the change of co-ordinates, also

$$= \int_{-1}^{1} \int_{0}^{2\pi} \frac{(1 - c^2)\, dpd\psi}{(1 + c^2 - 2cp)^{\frac{3}{2}}},$$

which can be at once integrated. It

$$= 2\pi (1 - c^2) \int_{-1}^{1} \frac{dp}{(1 + c^2 - 2cp)^{\frac{3}{2}}}$$

$$= 2\pi \frac{1 - c^2}{c} \frac{1}{\sqrt{1 + c^2 - 2cp}} + \text{const.}$$

$$= 2\pi \frac{1 - c^2}{c} \left(\frac{1}{1 - c} - \frac{1}{1 + c} \right)$$

$$= 2\pi \left(\frac{1 + c}{c} - \frac{1 - c}{c} \right)$$

$$= 4\pi.$$

It is remarkable that this result is altogether independent of c*.

* This result can be obtained more shortly as follows ; but the proof given

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31. By analysing the integral in the last Article and separating it into its elements, we can show by what process c vanishes from the result, and this will assist us in the latter part of the present demonstration. It matters not in which order we effect the integration; we shall therefore integrate with respect to p first, because it becomes necessary to do so in the next Article.

The quantity $1-p$ is the versed-sine of the arc QP, and is measured along the line QCq. Let this line be divided into n parts each equal to dp, so that $n \cdot dp =$ the diameter $= 2$, n being very large and dp very small. Draw perpendiculars to the diameter through these divisions cutting the circle QPq in a series of points; and call the distances of these points from D, beginning from Q,

$$1-c,\ s',\ s'',\ s'''\ldots\ldots s^{(n-1)},\ 1+c.$$

Suppose P is at the xth division; then by expansion, omitting the squares and higher powers of dp as they vanish in the limit with reference to the first power, we see the truth of the following:

$$\frac{1}{s^{(x)}} - \frac{1}{s^{(x+1)}} = \frac{1}{\sqrt{1+c^2-2cp}} - \frac{1}{\sqrt{1+c^2-2c\,(p+dp)}}$$

$$= \frac{-cdp}{(1+c^2-2cp)^{\frac{3}{2}}}\ *.$$

in the text is necessary for comprehending the remainder of our demonstration. The equivalent of the fraction to be integrated is $(1+3P_1c+5P_2c^2+\ldots)\,d\mu' d\omega'$. The property of Laplace's coefficients proved in Art. 27, shows that every term of this series except the first will vanish in the integration, and the first will give 4π.

* This formula may be proved *geometrically* thus. Draw a diagram according to the following description. On Qq as a diameter describe a semicircle $QPpq$, p being very near P, and C the centre. Take D in Cq, so that $CD=c$. Join DP, Dp, CP. Draw PM, pm perpendicular to Qq, Pn to pm, and pr to DP. Join nr.

Then because the angles at n and r are right angles, a circle can be drawn through P, p, n, r;

$\therefore$ angle Ppr = angle Pnr, or angle CPD = angle Pnr;

also angle nPr = angle PDC;

$\therefore$ rPn and CDP are similar triangles;

$$\therefore\ \frac{Pn}{Pr} = \frac{DP}{DC},\ \text{or}\ \frac{DC \cdot Mm}{DP} = DP - Dp\ \text{ultimately};$$

By giving x its successive values from 0 to $n-1$, and adding together all the resulting values of the above expression, and afterwards taking the limit, we have the definite integral with respect to p; the limits are from $p=1$ to $p=-1$, or, changing the sign, from $p=-1$ to $p=1$. Thus, n being made infinitely great,

$$\int_{-1}^{1} \frac{c\,dp}{(1+c^2-2cp)^{\frac{3}{2}}} = \left(\frac{1}{1-c} - \frac{1}{s'}\right) + \left(\frac{1}{s'} - \frac{1}{s''}\right) + \ldots$$

$$\ldots + \left(\frac{1}{s^{(n-1)}} - \frac{1}{1+c}\right).$$

Hence,

$$\int_0^{2\pi}\int_{-1}^{1} \frac{(1-c^2)\,d\psi\,dp}{(1+c^2-2cp)^{\frac{3}{2}}}$$

$$= \int_0^{2\pi} d\psi \left[\frac{1+c}{c} + \frac{1-c^2}{c}\left\{\left(\frac{1}{s'} - \frac{1}{s'}\right) + \ldots\right.\right.$$

$$\left.\left. \ldots + \left(\frac{1}{s^{(n-1)}} - \frac{1}{s^{(n-1)}}\right)\right\} - \frac{1-c}{c}\right]$$

$$= 2\pi\left(\frac{1+c}{c} - \frac{1-c}{c}\right) = 4\pi, \text{ as before.}$$

Here it will be seen that the terms within the inner brackets mutually destroy each other whatever be the value of c. It may also be observed that were this not the case, that whole part of the expression would vanish for the particular value $c=1$ (which is the only case we shall have to use), whatever the value of the sum of the terms following the multiplier $1-c^2$, so long as that sum is not infinite. This leads us on to the last stage of the demonstration.

32. We have now to show that when $c=1$,

$$\int_{-1}^{1}\int_0^{2\pi} (1-c^2)\,\frac{F(\mu', \omega')\,d\mu'\,d\omega'}{(1+c^2-2cp)^{\frac{3}{2}}} = 4\pi F(\mu, \omega).$$

$$\therefore \frac{DC \,.\, Mm}{DP^3} = \frac{Dp}{DP}\left(\frac{1}{Dp} - \frac{1}{DP}\right) = \frac{1}{Dp} - \frac{1}{DP} \text{ ultimately,}$$

which is the formula in the text. The whole proof of this fundamental principle of Laplace's Functions may therefore be conducted geometrically, as the remainder is already in that form.

The function $F(\mu', \omega')$ at the point Q is $F(\mu, \omega)$, call it F: let $F', F'' \ldots F^{(n)}$ be its values at the points of junction of the successive elements along the great circle QPq. Then by multiplying the successive values of $\frac{1}{s^{(x)}} - \frac{1}{s^{(x+1)}}$ by $F, F', F'' \ldots$ dividing them by c, and adding them together, we have

$$\int_{-1}^{1}\int_{0}^{2\pi} (1-c^2)\frac{F(\mu', \omega')\, d\mu' d\omega'}{(1+c^2-2cp)^{\frac{3}{2}}},$$

$$\text{or } \int_{0}^{2\pi}\int_{-1}^{1} (1-c^2)\frac{F(\mu', \omega') d\psi\, dp}{(1+c^2-2cp)^{\frac{3}{2}}},$$

$$= \int_0^{2\pi} d\psi \frac{1-c^2}{c}\left\{\left(\frac{1}{1-c} - \frac{1}{s'}\right)F + \left(\frac{1}{s'} - \frac{1}{s''}\right)F' + \ldots + \left(\frac{1}{s^{(n-1)}} - \frac{1}{1+c}\right)F^{(n-1)}\right\}$$

$$= \int_0^{2\pi} d\psi \left[\frac{1+c}{c}F + \frac{1-c^2}{c}\left\{\left(\frac{F''}{s'} - \frac{F}{s'}\right) + \left(\frac{F'''}{s''} - \frac{F'}{s''}\right) + \ldots\right\} - \frac{1-c}{c}F^{(n-1)}\right]$$

$$= \int_0^{2\pi} d\psi \left[\frac{1+c}{c}F + \frac{1+c}{c}\left\{(F'-F)\frac{1-c}{s'} + (F''-F')\frac{1-c}{s''} + \ldots\right\} - \frac{1-c}{c}F^{(n-1)}\right]$$

n being made infinitely great.

The fractions $\frac{1-c}{s'}, \frac{1-c}{s''}, \ldots$ diminish successively in value, being the ratios of QD to the successive values of DP. When $c = 1$ each of them vanishes; and none of the factors $F'-F, F''-F', \ldots$ become infinite. Hence the expression to be integrated becomes $d\psi \,.\, 2F(\mu, \omega)$ when $c = 1$, and the integral of it is $4\pi F(\mu, \omega)$, since μ and ω are altogether independent of ψ. Equating this and the series which represents this same integral,

$$4\pi F(\mu, \omega) = \int_{-1}^{1}\int_{0}^{2\pi} \{1 + 3P_1 + \ldots + (2i+1)P_i + \ldots\} F(\mu', \omega') d\mu' d\omega';$$

$$\therefore\ F(\mu, \omega) = \int_{-1}^{1}\int_{0}^{2\pi} \left\{\frac{F(\mu', \omega')}{4\pi} + \frac{3F(\mu', \omega')}{4\pi} P_1 + \ldots\ldots \right.$$

$$\left. + \frac{(2i+1)\, F(\mu', \omega')}{4\pi} P_i + \ldots\ldots\right\} d\mu' d\omega'.$$

The general term of this, viz.

$$\int_{-1}^{1}\int_{0}^{2\pi} \frac{2i+1}{4\pi} F(\mu', \omega') P_i d\mu' d\omega',$$

which we will call F_i, is a function of μ and ω; and evidently satisfies Laplace's Equation in μ and ω, because P_i does so. Hence, this is a Laplace's Function, of the ith order: and the result is, what we were to demonstrate, that any function of μ and ω may be expanded in a series of Laplace's Functions; or,

$$F(\mu, \omega) = F_0 + F_1 + F_2 + \ldots\ldots + F_i + \ldots\ldots$$

33. Those who are at all acquainted with the controversy which followed the first discovery of these remarkable functions by Laplace, will understand why we have entered so fully upon the subject. Laplace's demonstration in the *Mécanique Céleste* was by no means conclusive. This Mr Ivory pointed out in the *Philosophical Transactions* for 1812; and in the Volume for 1822 he threw considerable doubt upon the applicability of the theorem to functions that are not rational and entire functions of μ, $\sqrt{1-\mu^2}\cos\omega$, $\sqrt{1-\mu^2}\sin\omega$. Poisson wrote much upon the subject. In the first edition of the author's *Mechanical Philosophy* the last method of Poisson was followed, as given in his *Théorie Mathématique de la Chaleur;* in which he effects the integration of the fraction on the left-hand side by the artifice of substituting for it an integrable, but entirely different fraction in its general form, but which coincides with it in the particular case for which he requires it in the result, viz. when $c = 1$. In the Second Edition of the *Mechanical Philosophy* we gave a much shorter proof, based upon an idea taken from Professor O'Brien's *Mathematical Tracts*. But this also rather concealed the real difficulty of the case, and passed it over by an artifice. In the demonstration now given, we have gone to the foundation of the calculus, the doctrine of limits, and attempted to clear up all difficulty and ambiguity in the matter.

With regard to the doubt thrown out by Ivory, alluded to above, it seems to be clear that theoretically every function can be expanded in a series of Laplace's Functions: but if it be not a rational function of the co-ordinates, the number of

terms in the series will be infinite, and if the terms be not convergent, the expansion, or rather arrangement, will be useless. But this must be determined in each case. A similar uncertainty, requiring examination, always attends the use of infinite series.

PROP. *To prove that a function of μ and ω can be arranged in only one series of Laplace's Functions.*

34. For if possible let both these be true,

$$F(\mu, \omega) = F_0 + F_1 + F_2 + \ldots\ldots + F_i + \ldots\ldots$$

$$F(\mu, \omega) = G_0 + G_1 + G_2 + \ldots\ldots + G_i + \ldots\ldots$$

$$\therefore\ 0 = F_0 - G_0 + (F_1 - G_1) + \ldots\ldots + (F_i - G_i) + \ldots\ldots$$

and if these letters be accented when μ' and ω' are the variables instead of μ and ω, then

$$0 = (F_0' - G_0') + (F_1' - G_1') + \ldots\ldots + (F_i' - G_i') + \ldots\ldots$$

$$\therefore\ 0 = \int_{-1}^{1}\int_{0}^{2\pi} P_i (F_i' - G_i')\, d\mu' d\omega', \text{ by Art. 27.}$$

But the principle demonstrated in the last Proposition shows that

$$F_i - G_i = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} (1 + 3P_1 + \ldots)(F_i' - G_i')\, d\mu' d\omega',$$

$$= \frac{2i+1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} P_i (F_i' - G_i')\, d\mu' d\omega', \text{ by Art. 27,}$$

$$= 0, \text{ by the condition deduced above;}$$

therefore $F_i = G_i$, and the two series are term by term identical, and the Proposition is true.

35. It follows from this, that if by any process we can expand a function in a series of quantities which satisfy Laplace's Equation, that is the only series of the kind into which it can be expanded: and if by any other process we obtain what is apparently another, the terms of the two series must be the same, term by term, and we may put them equal to each other.

36. Before concluding this Chapter, we shall explain how the numerical coefficients in $P_0 P_1 \ldots P_i \ldots$ are found: and shall give a few examples of the truth of the last Proposition but one (that in Art. 32) by actual integration.

PROP. *To explain how to expand* P_i.

37. By Art. 25 P_i is the coefficient of c^i in the expansion of the function

$$[1 + c^2 - 2c\{\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega - \omega')\}]^{-\frac{1}{2}},$$

and is therefore a rational and entire function of μ,

$$\sqrt{1-\mu^2}\cos\omega, \text{ and } \sqrt{1-\mu^2}\sin\omega;$$

and is precisely the same function of μ',

$$\sqrt{1-\mu'^2}\cos\omega', \text{ and } \sqrt{1-\mu'^2}\sin\omega'.$$

The general term of P_i, viz. that involving $\cos n(\omega - \omega')$, can arise solely from the powers $n, n+2, n+4, \ldots$ of $\cos(\omega - \omega')$. Hence $(1-\mu^2)^{\frac{n}{2}}$ will occur as a factor of that term: and the other part of its coefficient will be a factor of the form

$$A_0\mu^{i-n} + A_1\mu^{i-n-2} + \ldots + A_s\mu^{i-n-2s} + \ldots = H_n \text{ suppose.}$$

Hence

$$P_i = H_0 + (1-\mu^2)^{\frac{1}{2}}H_1\cos(\omega-\omega') + \ldots + (1-\mu^2)^{\frac{n}{2}}H_n\cos n(\omega-\omega') + \ldots$$

If this be substituted for P_i in Laplace's Equation and the coefficient of $\cos n(\omega - \omega')$ be equated to zero, we obtain a condition from which to calculate the arbitrary constants we have introduced. This condition, after reduction and arrangement, is as follows:

$$0 = (i-n)(i+n+1)H_n(1-\mu^2)^n + \frac{d}{d\mu}\left\{(1-\mu^2)^{n+1}\frac{dH_n}{d\mu}\right\}.$$

Substituting in this the series which H_n represents, and equating the coefficient of the general term $(1-\mu^2)^n\mu^{i-n-2s}$ to zero, and reducing, we arrive at the formula

$$A_s = -\frac{(i-n-2s+2)(i-n-2s+1)}{2s(2i-2s+1)}A_{s-1}.$$

By making s successively equal 1, 2, 3 ... we have $A_1 A_2 \ldots$ in terms of A_0. Let these be substituted, and we have the coefficient of $\cos n(\omega - \omega') =$

$$A_0 (1-\mu^2)^{\frac{n}{2}} \left\{ \mu^{i-n} - \frac{(i-n)(i-n-1)}{2(2i-1)} \mu^{i-n-2} + \ldots\ldots \right\},$$

call this $A_0 f(\mu)$. The coefficient A_0 is a function of μ', but is independent of μ: and because P_i is the same function of μ' that it is of μ, it follows that $A_0 = \alpha_n f(\mu')$, where α_n is a numerical quantity: and the coefficient of

$$\cos n(\omega - \omega') = \alpha_n f(\mu') f(\mu).$$

To find α_n we must compare the first term of the ascending expansion of $\alpha_n f(\mu') f(\mu)$ in powers of μ with the corresponding term in the coefficient of c^i in the actual expansion of

$$[1 + c^2 - 2c\{\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega - \omega')\}]^{-\frac{1}{2}}.$$

This leads to the following result:

$$\alpha_n = 2 \left\{ \frac{1.3.5 \ldots (2i-1)}{1.2.3 \ldots i} \right\}^2 \frac{i(i-1)\ldots(i-n+1)}{(i+1)(i+2)\ldots(i+n)};$$

this applies when $n = 1, 2, 3 \ldots$, but evidently not when $n = 0$: α_0 is found by equating coefficients to be

$$\left\{ \frac{1.2.3 \ldots (2i-1)}{1.2.3 \ldots i} \right\}^2.$$

We have now the complete value of P_i in a series; it is as follows:

$$P_i = \left\{ \frac{1.3.5 \ldots (2i-1)}{1.2.3 \ldots i} \right\}^2$$

$$\times \left[\left\{ \mu^i - \frac{i(i-1)}{2(2i-1)} \mu^{i-2} + \frac{i(i-1)}{2(2i-1)} \frac{(i-2)(i-3)}{4(2i-3)} \mu^{i-4} - \&c.... \right\} \right.$$

$$\left. \times \left\{ \mu'^i - \frac{i(i-1)}{2(2i-1)} \mu'^{i-2} + \frac{i(i-1)}{2(2i-1)} \frac{(i-2)(i-3)}{4(2i-3)} \mu'^{i-4} - \&c.... \right\} \right]$$

$$+ 2\cos(\omega - \omega') \frac{i}{i+1}$$

$$\times(1-\mu^2)^{\frac{1}{2}}\left\{\mu^{i-1}-\frac{(i-1)(i-2)}{2(2i-1)}\mu^{i-3}+\frac{(i-1)(i-2)}{2(2i-1)}\frac{(i-3)(i-4)}{4(2i-3)}\mu^{i-5}\ldots\right\}$$

$$\times(1-\mu'^2)^{\frac{1}{2}}\left\{\mu'^{i-1}-\frac{(i-1)(i-2)}{2(2i-1)}\mu'^{i-3}+\frac{(i-1)(i-2)}{2(2i-1)}\frac{(i-3)(i-4)}{4(2i-3)}\mu'^{i-5}\ldots\right\}$$

$$+2\cos 2(\omega-\omega')\frac{i(i-1)}{(i+1)(i+2)}$$

$$\times(1-\mu^2)\left\{\mu^{i-2}-\frac{(i-2)(i-3)}{2(2i-1)}\mu^{i-4}+\frac{(i-2)(i-3)}{2(2i-1)}\frac{(i-4)(i-5)}{4(2i-3)}\mu^{i-6}\ldots\right\}$$

$$\times(1-\mu'^2)\left\{\mu'^{i-2}-\frac{(i-2)(i-3)}{2(2i-1)}\mu'^{i-4}+\frac{(i-2)(i-3)}{2(2i-1)}\frac{(i-4)(i-5)}{4(2i-3)}\mu'^{i-6}\ldots\right\}$$

$$+\&c.\ldots.\Big].$$

38. The following numerical examples are written down for convenience of reference:

(1) $P_1=\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega')$.

(2) $$P_2=\frac{9}{4}\left\{\left(\mu^2-\frac{1}{3}\right)\left(\mu'^2-\frac{1}{3}\right)+\frac{4}{3}(1-\mu^2)^{\frac{1}{2}}\mu(1-\mu'^2)^{\frac{1}{2}}\mu'\cos(\omega-\omega')\right.$$
$$\left.+\frac{1}{3}(1-\mu^2)(1-\mu'^2)\cos 2(\omega-\omega')\right\}.$$

(3) $$P_3=\frac{25}{4}\left\{(\mu^3-\frac{3}{5}\mu)(\mu'^3-\frac{3}{5}\mu')\right.$$
$$+\frac{3}{2}(1-\mu^2)^{\frac{1}{2}}\left(\mu^2-\frac{1}{5}\right)(1-\mu'^2)^{\frac{1}{2}}\left(\mu'^2-\frac{1}{5}\right)\cos(\omega-\omega')$$
$$\left.+\frac{3}{5}(1-\mu^2)\mu(1-\mu'^2)\mu'\cos 2(\omega-\omega')+\frac{1}{10}(1-\mu^2)^{\frac{3}{2}}(1-\mu'^2)^{\frac{3}{2}}\cos 3(\omega-\omega')\right\}$$
$$\&c.=\&c.$$

39. The following are some examples of expanding a function in a series of Laplace's Functions, by an application of the formula

$$F_i = \frac{2i+1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} F(\mu', \omega')\, P_i d\mu' d\omega',$$

proved in Art. 32.

Ex. 1. Arrange $a + b\mu^2$ in terms of Laplace's Functions.

Here $F(\mu', \omega') = a + b\mu'^2$. First put $i = 0$, $P_0 = 1$;

$$\therefore F_0 = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}(a + b\mu'^2)\, d\mu'\, d\omega' = \frac{1}{2}\int_{-1}^{1}(a + b\mu'^2)\, d\mu'$$

$$= \frac{1}{2}(a\mu' + \frac{1}{3}b\mu'^3 + \text{const.}) = a + \frac{1}{3}b.$$

Again, put $i = 1$, P_1 is found in the last Article.

$$\therefore F_1 = \frac{3}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}(a+b\mu'^2)\{\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega')\}\, d\mu'\, d\omega'$$

$$= \frac{3}{4\pi}\int_{-1}^{1}(a+b\mu'^2)\{\mu\mu'.\omega' - \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\sin(\omega-\omega')\}\, d\mu',$$

between the proper limits, $\omega' = 0$ and $\omega' = 2\pi$,

$$= \frac{3}{2}\int_{-1}^{1}(a+b\mu'^2)\mu\mu' d\mu' = \frac{3}{2}\mu\left(\frac{1}{2}a\mu'^2 + \frac{1}{4}b\mu'^4\right),$$

between the limits $\mu' = -1$ and $\mu' = 1, = 0$.

Next, put $i = 2$, and substitute for P_2 from the last Article.

$$\therefore F_2 = \frac{5}{4\pi}\int_{-1}^{1}\int_{0}^{2}(a+b\mu'^2)\left\{\frac{9}{4}\left(\mu^2-\frac{1}{3}\right)\left(\mu'^2-\frac{1}{3}\right) + M\cos(\omega-\omega') + N\cos 2(\pi-\omega')\right\} d\mu'\, d\omega'$$

$$= \frac{5}{2}\int_{-1}^{1}(a+b\mu'^2)\frac{9}{4}\left(\mu^2-\frac{1}{3}\right)\left(\mu'^2-\frac{1}{3}\right) d\mu'$$

$$= \frac{45}{8}\left(\mu^2-\frac{1}{3}\right)\int_{-1}^{1}\left\{-\frac{1}{3}a + (a-\frac{1}{3}b)\mu'^2 + b\mu'^4\right\} d\mu'$$

$$= \frac{45}{8}\left(\mu^2-\frac{1}{3}\right)\left\{-\frac{2}{3}a + \frac{2}{3}(a-\frac{1}{3}b) + \frac{2}{5}b\right\} = b\left(\mu^2-\frac{1}{3}\right)$$

Hence the function $a + b\mu^2$ stands as follows, when arranged in terms of Laplace's Functions,

$$(a + \frac{1}{3}b) + b\left(\mu^2 - \frac{1}{3}\right),$$

and consists of two Functions, of the order 0 and 2 respectively. The above is a long process to arrive at this result. It might have been so arranged at a glance. But the calculation has been given as an example of the use of the formula, which in most cases is the only means of obtaining the desired result.

Ex. 2. Arrange $49 + 30\mu + 3\mu^2 + \sqrt{1-\mu^2}\,(40+72\mu)\cos(\omega-\alpha)$ $+ 24\,(1-\mu^2)\cos 2\,(\omega-\alpha)$ in terms of Laplace's Functions.

The result is $50 + \{30\mu + 40\sqrt{1-\mu^2}\cos(\omega-\alpha)\}$ $+ \{3\mu^2 - 1 + 72\mu\sqrt{1-\mu^2}\cos(\omega-\alpha) + 24\,(1-\mu^2)\cos 2\,(\omega-\alpha)\}$, consisting of three functions of the orders, 0, 1, 2.

Ex. 3. Let the function be

$$1 + \sqrt{2-2\mu^2}\cos(\omega+\alpha) + \frac{1}{2}(1-\mu^2)\cos 2\,(\omega+\alpha).$$

The first term is a Laplace's Function of the order 0, and the second and third terms taken together are one of the second order.

Ex. 4. Let $1 - (1-\mu^2)\cos^2\omega$ be the function. The arrangement is

$$\frac{2}{3} + \frac{1}{2}\left\{\left(\mu^2 - \frac{1}{3}\right) - (1-\mu^2)\cos 2\omega\right\},$$

or, which is the same,

$$\frac{2}{3} + \left\{\frac{1}{3} - (1-\mu^2)\cos^2\omega\right\}.$$

CHAPTER III.

ATTRACTION OF BODIES NEARLY SPHERICAL.

40. As the Earth and other bodies of the Solar System are nearly spherical, and yet may not be precisely of the spheroidal form, it is found necessary in questions of Physical Astronomy to calculate the attraction of bodies nearly spherical. In these calculations is seen the value of the Functions we have been considering in the last Chapter.

If $r'\theta'\omega'$ be the co-ordinates to any element of the attracting mass, ρ' be its density, and $\cos\theta' = \mu'$, then the mass of this element

$$= \rho' dr' r' d\theta' r' \sin\theta' d\omega' = -\rho' r'^2 dr' d\mu' d\omega',$$

and the reciprocal of the distance being R, by Art. 18 and 25, the potential V

$$= \int_0^r \int_{-1}^{1} \int_0^{2\pi} \rho' \left(P_0 \frac{r'^2}{r} + P_1 \frac{r'^3}{r^2} + \ldots + P_i \frac{r'^{i+2}}{r^{i-1}} + \ldots \right) dr' d\mu' d\omega';$$

$$\text{or } \int_0^r \int_{-1}^{1} \int_0^{2\pi} \rho' \left(P_0 r' + P_1 r + P_2 \frac{r^2}{r'} + \ldots + P_i \frac{r^i}{r'^{i-1}} + \ldots \right) dr' d\mu' d\omega',$$

according as r, the distance of the attracted point from the origin, is greater or less than r'. We shall proceed soon to use these formulæ; but we must first find the value of V for a perfect sphere.

PROP. *To calculate the value of V for a homogeneous sphere.*

41. Let the centre of the sphere be the origin of the polar co-ordinates $(r'\mu'\omega')$ to any element of its mass, and the line through the attracted point be that from which the angles are

measured, and ρ the density. Then $-\rho r'^2 dr'\, d\mu'\, d\omega'$ is the mass of the element: its distance from the attracted point

$$= \sqrt{r^2 + r'^2 - 2rr'\cos\omega}.$$

Hence, a being the radius of the sphere,

$$V = \int_0^a \int_{-1}^{1} \int_0^{2\pi} \frac{\rho r'^2\, dr'\, d\mu'\, d\omega'}{\sqrt{r^2 + r'^2 - 2rr'\mu'}}$$

$$= 2\pi\rho \int_0^a \int_{-1}^{1} \frac{r'^2\, dr'\, d\mu'}{\sqrt{r^2 + r'^2 - 2rr'\mu'}} = 2\pi\rho \int_0^a -\frac{r'}{r} \left\{ \sqrt{r^2 + r'^2 - 2rr'\mu'} \right\} dr',$$

from $\mu' = -1$ to $\mu' = 1$, $= 2\pi\rho \int_0^a \frac{r'}{r} \left\{ (r + r') \mp (r - r') \right\} dr'$,

$-$ when the attracted point is without, and $+$ when it is within the shell,

$$= \frac{4\pi\rho}{r} \int_0^a r'^2 dr' = \frac{4\pi\rho a^3}{3r},$$

when the point is without the sphere.

When the point is within the sphere, the part of V for the shells which enclose the point

$$= 2\pi\rho \int_r^a 2r'\, dr' = 2\pi\rho\,(a^2 - r^2):$$

and the part of V for the other shells of the sphere

$$= \frac{4\pi\rho}{r} \int_0^r r'^2 dr' = \frac{4}{3}\pi\rho r^2.$$

Hence $V = \frac{4\pi\rho a^3}{3r}$ for an *external* particle,

$$V = 2\pi\rho a^2 - \frac{2}{3}\pi\rho r^2 \text{ for an internal particle.}$$

PROP. *To find the attraction of a homogeneous body, differing little from a sphere in form, on a particle without it.*

42. Since the attracted particle is without the attracting mass, we must expand V in a descending series of powers of r, and shall therefore use the first of the expressions for V in Art. 40. Let the mean radius of the body $= a$; and let $a(1+y')$ be the variable radius, y' being a function of μ' and ω', and its square being neglected.

Then, for the excess of the attracting mass over the sphere of which the radius $= a$, effecting the integration with respect to r' from $r' = a$ to $r' = a(1+y')$, the value of V

$$= \rho \int_{-1}^{1} \int_{0}^{2\pi} \left\{ \frac{a^3}{r} P_0 + \frac{a^4}{r^2} P_1 + \ldots + \frac{a^{i+3}}{r^{i+1}} P_i + \ldots \right\} y' \, d\mu' d\omega'.$$

But if y, the same function of μ and ω that y' is of μ' and ω', be expanded in a series of Laplace's Functions, viz.

$$Y_0 + Y_1 + \ldots + Y_i + \ldots,$$

then the theorems of Art. 27 and 32 show that

$$\frac{2i+1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} y' P_i \, d\mu' d\omega' = \frac{2i+1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} Y_i' P_i \, d\mu' d\omega' = Y_i.$$

Hence the value of V for the excess over the sphere becomes

$$= \frac{4\pi\rho a^3}{r} \left\{ Y_0 + \frac{a}{3r} Y_1 + \ldots + \frac{a^i}{(2i+1) r^i} Y_i + \ldots \right\};$$

and the part of V for the sphere, rad. $= a$, is

$$V = \frac{4\pi\rho a^3}{3r}.$$

Hence for the whole mass

$$V = \frac{4\pi\rho a^3}{3r} + \frac{4\pi\rho a^3}{r} \left\{ Y_0 + \frac{a}{3r} Y_1 + \ldots + \frac{a^i}{(2i+1) r^i} Y_i + \ldots \right\}.$$

This is the first example in which we see the great value of the properties of Laplace's Functions; they here give us at

once the integrals involved in our expression for V, in terms of the equation to the surface of the attracting mass, without integration.

From the expression for V the attraction can be immediately found by the formula of Art. 20. Thus

$$\text{attraction} = -\frac{dV}{dr}$$

$$= \frac{4\pi\rho a^3}{3r^2} + \frac{4\pi\rho a^3}{r^2}\left\{ Y_0 + \frac{2a}{3r} Y_1 + \ldots \frac{(i+1)\,a^i}{(2i+1)\,r^i} Y_i + \ldots \right\}.$$

PROP. *To find the attraction of a homogeneous body, differing but little from a sphere, on a particle within its mass.*

43. We must in this case expand V in an ascending series of powers of r; and shall therefore take the second of the series of Art. 40. By proceeding as in the last Proposition, we find that the part of V which appertains to the excess over the sphere

$$= \rho \int_{-1}^{1}\int_{0}^{2\pi} \left\{ a^2 P_0 + arP_1 + \ldots + \frac{r^i}{a^{i-2}} P_i + \ldots \right\} y' d\mu' d\omega',$$

or $$= 4\pi\rho a^2 \left\{ Y_0 + \frac{r}{3a} Y_1 + \ldots + \frac{r^i}{(2i+1)\,a^i} Y_i + \ldots \right\}.$$

Adding to this the part of V which appertains to the sphere of radius a, viz. $2\pi\rho a^2 - \frac{2}{3}\pi\rho r^2$, for the whole mass,

$$V = 2\pi\rho a^2 - \frac{2}{3}\pi\rho r^2 + 4\pi\rho a^2 \left\{ Y_0 + \frac{r}{3a} Y_1 + \ldots + \frac{r^i}{(2i+1)\,a^i} Y_i + \ldots \right\},$$

And the attraction $= -\dfrac{dV}{dr}$

$$= \frac{4\pi}{3}\rho r - 4\pi\rho a \left\{ \frac{1}{3} Y_1 + \frac{2r}{5a} Y_2 + \ldots + \frac{ir^{i-1}}{(2i+1)\,a^{i-1}} Y_i + \ldots \right\}.$$

We can show that by properly choosing the value of (a) and the origin of the radius of the surface we can make Y_0 and Y_1 disappear from the above formulæ.

PROP. *To show that by choosing* a *equal to the radius of the sphere of which the mass equals that of the attracting body we cause* Y_0 *to vanish, and by taking the centre of gravity of the body as the origin of the radius vector, we cause* Y_1 *to vanish.*

44. The mass of the body

$$= \rho \int_0^r \int_{-1}^1 \int_0^{2\pi} r^2 dr\, d\mu\, d\omega = \frac{1}{3}\rho \int_{-1}^1 \int_0^{\pi} \mathrm{r}^3 d\mu\, d\omega,$$

where r is the radius vector of the surface of the body, and $= \mathrm{a}(1+y)$ suppose. Putting this for r, the mass of the body

$$= \text{mass of sphere (rad.} = \mathrm{a}) + \rho \mathrm{a}^3 \int_{-1}^1 \int_0^{2\pi} y d\mu\, d\omega$$

$$= \text{mass of sphere} + \rho \mathrm{a}^3 \int_{-1}^1 \int_0^{2\pi} Y_0, \text{ by Art. 27,}$$

$$= \text{mass of sphere} + 4\pi\rho \mathrm{a}^3 Y_0.$$

If then a be taken equal to the radius of the sphere of which the mass equals the mass of the body, $Y_0 = 0$, as was stated.

Again, let $\bar{x}\bar{y}\bar{z}$ be the co-ordinates to the centre of gravity of the body, M its mass: the co-ordinates to the element of which the mass is $-\rho r^2 dr\, d\mu\, d\omega$ are

$$r\sqrt{1-\mu^2}\cos\omega, \quad r\sqrt{1-\mu^2}\sin\omega, \quad \text{and } r\mu;$$

$$\therefore M.\bar{x} = \int_0^r \int_{-1}^1 \int_0^{2\pi} \rho r^3 \sqrt{1-\mu^2}\cos\omega\, dr\, d\mu\, d\omega$$

$$= \frac{1}{4}\int_{-1}^1 \int_0^{2\pi} \rho \mathrm{r}^4 \sqrt{1-\mu^2}\cos\omega\, d\mu\, d\omega,$$

$$M.\bar{y} = \int_0^r \int_{-1}^1 \int_0^{2\pi} \rho r^3 \sqrt{1-\mu^2}\sin\omega\, dr\, d\mu\, d\omega$$

$$= \frac{1}{4}\int_{-1}^1 \int_0^{2\pi} \rho \mathrm{r}^4 \sqrt{1-\mu^2}\sin\omega\, d\mu\, d\omega,$$

$$M.\bar{z} = \int_0^r \int_{-1}^{1} \int_0^{2\pi} \rho r^3 \mu \, dr \, d\mu \, d\omega = \frac{1}{4} \int_{-1}^{1} \int_0^{2\pi} r^4 \mu \, d\mu \, d\omega ;$$

putting $r = a(1+y) = a(1 + Y_0 + Y_1 + \ldots + Y_i \ldots)$, and observing that $\sqrt{1-\mu^2}\cos\omega$, $\sqrt{1-\mu^2}\sin\omega$, and μ satisfy Laplace's Equation, and are of the first order, we have by Art. 27,

$$M.\bar{x} = \rho a^4 \int_{-1}^{1} \int_0^{2\pi} Y_1 \sqrt{1-\mu^2} \cos\omega \, d\mu \, d\omega,$$

$$M.\bar{y} = \rho a^4 \int_{-1}^{1} \int_0^{2\pi} Y_1 \sqrt{1-\mu^2} \sin\omega \, d\mu \, d\omega,$$

$$M.\bar{z} = \rho a^4 \int_{-1}^{1} \int_0^{2\pi} Y_1 \mu \, d\mu \, d\omega.$$

But Y_1, being a function of μ, $\sqrt{1-\mu^2}\cos\omega$, and $\sqrt{1-\mu^2}\sin\omega$ of the first order, is of the form

$$A\sqrt{1-\mu^2}\cos\omega + B\sqrt{1-\mu^2}\sin\omega + C\mu ;$$

$$\therefore M.\bar{x} = \frac{4}{3}\pi\rho a^4 A, \quad M.\bar{y} = \frac{4}{3}\pi\rho a^4 B, \quad M.\bar{z} = \frac{4}{3}\pi\rho a^4 C.$$

Hence if we take the origin of co-ordinates at the centre of gravity and therefore $\bar{x}=0$, $\bar{y}=0$, $\bar{z}=0$, we have $A=0$, $B=0$, $C=0$, and therefore $Y_1 = 0$, as stated in the enunciation.

PROP. *To find the attraction of a heterogeneous body upon a particle without it; the body consisting of thin strata nearly spherical, homogeneous in themselves, but differing one from another in density.*

45. Let $a'(1+y')$ be the radius of the external surface of any stratum, a' being chosen so that

$$y' = Y_1' + Y_2' + \ldots + Y_i' + \ldots \text{ (Art. 44)}.$$

Since the strata are supposed not to be similar to one another, y' is a function of a' as well as of μ' and ω'. Let ρ' be the density of the stratum of which the mean radius is a'. Now the value of V for this stratum equals the difference between the values of V for two homogeneous bodies of

the density ρ' and mean radii a' and $a' - da'$. But for the body of which the mean radius is a' (Art. 42)

$$V = \frac{4\pi\rho' a'^3}{3r} + \frac{4\pi\rho' a'^3}{r}\left\{\frac{a'}{3r}\,Y_1' + \ldots + \frac{a'^i}{(2i+1)\,r^i}\,Y_i' + \ldots\right\}.$$

Hence for the stratum of which the external mean radius is a',

$$V = \frac{4\pi\rho' a'^2}{r}\,da' + \frac{4\pi\rho'}{r}\,\frac{d}{da'}\left\{\frac{a'^4}{3r}\,Y_1' + \ldots + \frac{a'^{i+3}}{(2i+1)\,r^i}\,Y_i' + \ldots\right\}da',$$

and therefore for the whole body,

$$V = \frac{4\pi}{r}\int_0^a \rho'\left\{a'^2 + \frac{d}{da'}\left(\frac{a'^4}{3r}\,Y_1' + \ldots + \frac{a'^{i+3}}{(2i+1)\,r^i}\,Y_i' + \ldots\right)\right\}da'.$$

From which the attraction is easily deduced.

PROP. *To find the attraction of the same body on an internal particle.*

46. Let $r = a\,(1+y)$ be the radius of the stratum in which the attracted particle lies. Then for the strata within the surface of which the radius is $a\,(1+y)$, we have

$$V = \frac{4\pi}{r}\int_0^a \rho'\left\{a'^2 + \frac{d}{da'}\left(\frac{a'^4}{3r}\,Y_1' + \ldots + \frac{a'^{i+3}}{(2i+1)\,r^i}\,Y_i' + \ldots\right)\right\}da'.$$

But for a stratum external to the particle we have by Art. 43,

$$V = 4\pi\rho' a'\,da' + 4\pi\rho'\,\frac{d}{da'}\left\{\frac{ra'}{3}\,Y_1' + \ldots + \frac{r}{(2i+1)\,a'^{i-2}}\,Y_i' + \ldots\right\}da'.$$

Consequently for the whole body,

$$V = \frac{4\pi}{r}\int_0^a \rho'\left\{a'^2 + \frac{d}{da'}\left(\frac{a'^4}{3r}\,Y_1' + \ldots + \frac{a'^{i+3}}{(2i+1)\,r^i}\,Y_i' + \ldots\right)\right\}da'$$

$$+ 4\pi\int_a^a \rho'\left\{a'\,da' + \frac{d}{da'}\left(\frac{ra'}{3}\,Y_1' + \ldots + \frac{r^i}{(2i+1)\,a'^{i-2}}\,Y_i' + \ldots\right)\right\}da'.$$

From this the attraction is readily obtained by differentiating with respect to r.

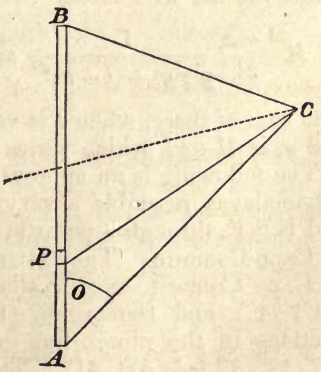
CHAPTER IV.

ATTRACTION OF TABLE-LANDS, MOUNTAINS, OCEANS, &c.

47. THE methods which have hitherto been given enable us to find the attraction of the Earth and other bodies of our system considered as a whole. But, taking the Earth as our example, the surface is irregular and does not follow any exact law of form. We ought, therefore, to be able to calculate the effect of these irregularities, and with this view the present Chapter is added to what has gone before. High table-lands may very materially affect the position of the plumb-line in some places. Enormous irregular mountain masses, like the Himmalayas, may do the same. Their effect ought, therefore, to be carefully estimated, as all instruments which are fixed by the plumb-line or spirit-level must be affected by such irregularities.

PROP. *To find the attraction of a slender prism of matter on a point in the line drawn to one of its extremities.*

48. Let AB be the prism, C the attracted point, P any element of the prism, $AP = r$, M the mass and l the length of the prism, $AC = a$, $BC = b$, $PC = y$, angle $PAC = \theta$.



Then the mass of the element at $P = M\dfrac{dr}{l}$.

Attraction of element at P on $C = M\dfrac{dr}{l}\dfrac{1}{y^2}$.

Ditto in direction CA

$$= M\frac{dr}{l}\frac{1}{y^2}\cos PCA.$$

$$= \frac{Mdr}{l} \cdot \frac{a - r\cos\theta}{y^3}, \quad y^2 = a^2 + r^2 - 2ar\cos\theta,$$

$$= \frac{Mdr}{l} \cdot \frac{a\sin^2\theta - \cos\theta\,(r - a\cos\theta)}{\{a^2 + r^2 - 2ar\cos\theta\}^{\frac{3}{2}}};$$

$\therefore$ attraction of whole prism

$$= \frac{M}{l} \frac{r - a\cos\theta + a\cos\theta}{a\sqrt{a^2 + r^2 - 2ar\cos\theta}}, \text{ from } r = 0 \text{ to } r = l,$$

$$= \frac{M}{al} \frac{l}{\sqrt{a^2 + l^2 - 2al\cos\theta}} = \frac{M}{ab}.$$

As this is symmetrical with respect to a and b, it shows that the particle is attracted equally towards the two extremities of the prism; and that therefore the resultant attraction acts in a line bisecting the angle which the prism subtends at the attracted point.

49. Cor. A uniform bar of very great length attracts a point not far from its centre with a force varying inversely as the distance from the bar. For let xy be the co-ordinates to the point from the centre measured along the bar and at right angles to it: $2l$ the length of the bar, M its mass. The bar is divided into two parts by y, and they attract the point towards the bar with forces

$$M\frac{l-x}{2l}\frac{1}{y\sqrt{y^2 + (l-x)^2}}, \text{ and } M\frac{l+x}{2l}\frac{1}{y\sqrt{y^2 + (l+x)^2}}.$$

The sum of these, when l is very large in comparison with x and y, is $M \div yl$, which varies inversely as y.

The following is an approximate illustration of this. The Himmalayas resemble a very long prism running W.N.W. and E.S.E. through a point in latitude 33°30′ in the longitude of Cape Comorin. They attract three places in the meridian of Cape Comorin—viz. Kaliana (lat. 29°30′48″), Kalianpur (24°7′11″), and Damargida (18°3′15″), so as to produce deflections in the plumb-line in the meridian equal to about 28″, 12″, 7″ (see Art. 61). The deflections towards the prism or axis of the Himmalayas may be taken to bear the same

proportion to each other as those in the meridian. Now the distances of the three stations from the point where the axis crosses the meridian are 3°39′, 9°23′, 15°27′ or 219′, 563′, 927′; in the same proportion are the distances of the stations from the prism or axis. It will be found that the reciprocals of these numbers are not far from being in the same proportion as the deflections. If Kalianpur were removed 20′ north the comparison would be exact.

PROP. *To find the attraction of a slender pyramid of any form upon a particle at its vertex; and also of a frustum of the pyramid.*

50. Let l be the length of the pyramid, α the area of a transverse section at distance unity from the vertex; r the distance of any section; αr^2 is its area; ρ the density of the matter: then $\alpha r^2 \rho dr$ is the mass of an element of the pyramid, and this divided by r^2 is its attraction;

$$\therefore \text{ attraction of pyramid on vertex } = \int^{l} \alpha \rho \, dr = \alpha \rho l.$$

If d is the length of any frustum of the pyramid, and $l = l' + d$, then

attraction of pyramid, length l', $= \alpha \rho l'$;

$$\therefore \text{ attraction of frustum } = \alpha \rho d.$$

It is observable that this is quite independent of the distance of the frustum from the vertex; and therefore all portions of the pyramid of equal length, any where selected, attract the vertex equally.

COR. Suppose the angular *width* of the pyramid to be β and to remain constant, while the angular depth varies; and let k be the linear depth of the transverse section of the base; then $\beta l k$ is the area of the base; and the attraction of the whole pyramid on the vertex $= \rho \beta k$. Hence, all slender pyramids having the same angular width and the same linear depth at the base attract their vertex alike, whatever their lengths be: or, which is the same thing, the angular width being the same the attraction varies as the linear depth of the

base, and is independent of the length. Thus, suppose it is required to find the effect of the deficiency of matter in the sea on a place on the sea coast, the shore of which shelves gradually. By dividing the sea into slender horizontal pyramids, the attraction of the shelving portion of it can be calculated by knowing only the depth at the extremities of the pyramids without knowing their lengths.

PROP. *To find the attraction of an extensive circular plain of given depth or thickness upon a station above its middle point.*

51. Let t be the thickness or depth; h the height of the particle from the nearer surface, c the radius, r the radius of any intermediate elementary annulus of the attracting mass, z its depth. The several elements of this annulus of matter will attract the particle towards the plane equally. Hence attraction of the particle

$$= \int_0^c \int_0^t \frac{2\pi\rho r\,(h+z)\,dr\,dz}{\{r^2+(h+z)^2\}^{\frac{3}{2}}} = \int_0^c 2\pi\rho \left\{\text{const.} - \frac{r}{\sqrt{r^2+(h+z)^2}}\right\} dr$$

$$= 2\pi\rho \int_0^c \left\{\frac{r}{\sqrt{r^2+h^2}} - \frac{r}{\sqrt{r^2+(h+t)^2}}\right\} dr$$

$$= 2\pi\rho \left\{\sqrt{c^2+h^2} - h - \sqrt{c^2+(h+t)^2} + h + t\right\}$$

$$= 2\pi\rho \left[t - \sqrt{c^2+h^2}\left\{\left(1+\frac{2ht+t^2}{c^2+h^2}\right)^{\frac{1}{2}} - 1\right\}\right]$$

$$= 2\pi\rho t \left\{1 + \frac{h+\frac{1}{2}t}{\sqrt{c^2+h^2}} + \ldots\right\}.$$

52. If the plain be of infinite extent, the attraction equals $2\pi\rho t$; and this remarkable result is true, that it is independent of the distance from the plain. The same will be the case if the height of the station above the middle of the attracting mass below, that is, $h+\frac{1}{2}t$, be so small that it may be neglected in comparison with the distance of the station from the furthest limit of the plain. Thus, for example, suppose the

height of the station above the middle of the mass below, that is, $h + \frac{1}{2}t$, is $\frac{1}{2}$ a mile and c 10 miles. Then the second term within the brackets is less than 0·05, and the attraction is very much the same as if the plain were unlimited in extent.

53. If ρ is the density of rock, taken to be half the mean density of the earth, $g = \frac{8}{3}\pi\rho a$. Hence the attraction of an extensive plain $= 2\pi\rho t = \frac{3}{4}\frac{t}{a}g$. Suppose, owing to geological changes of level, a continent is lifted up above the mean surface of the earth through a space t. Then gravity at a station on the continent will be diminished from this cause by the amount

$$g\left\{1 - \left(1 + \frac{t}{a}\right)^{-2}\right\} = \frac{2t}{a}g \text{ nearly.}$$

But the attraction of the underlying mass of thickness t must be taken into account. Hence the real diminution of gravity by the upheaval will be $\frac{5}{4}\frac{t}{a}g$. The ratio of this to the correction for increase of distance = 0·625*.

If the station be at the height h above the level of the continent, then the diminution

$$= \left(\frac{2h}{a} + \frac{5}{4}\frac{t}{a}\right)g.$$

This correction will depend upon the kind of rock of which the continent is made, whether of a dense or light description. Thus also for a station at sea, like St Helena, the correction would be different for a similar height above a continent, as sea water is only half the density of rock. The importance of these considerations will be seen in Art. 91, when we come to consider the vibration of a pendulum as a measure of gravity.

* Dr Young takes the ratio of the density of the surface to the mean density to be 5 : 11. In this case the correction would be $58 \div 88 = 0{\cdot}66$. See *Phil. Trans.* 1819, p. 93.

54. Cor. The result of this Proposition when the plain is unlimited in extent might have been foreseen from the result in the previous Proposition regarding the attraction of the frustum of a pyramid. Conceive an infinite number of slender pyramids to be drawn from the station intersecting the attracting plain; they will cut out of it an equal number of frustra, and the cosines of the angles they make with the perpendicular to the plain will be the thickness divided by the lengths of the frustra. But the attractions of the frustra are proportional to their lengths, and independent of the distance from the attracted point: (see Art. 50). Hence the resultant attraction of the whole will depend solely upon the thickness or depth of matter constituting the plain.

Prop. *To find the attraction of a rectangular mass, of small elevation compared with its length and breadth, upon a point lying in the plane of one of its larger sides.*

55. Let the attracted point be the origin of co-ordinates; the axes of x and y parallel to the long edges of the tabular mass, the axis of z being measured upwards. Let $x'y'z'$ be the co-ordinates to any point of the mass: xy co-ordinates to the nearest angle, XY to the furthest angle, H the height of the mass; ρ the density, supposed the same throughout.

Then $\rho dx'dy'dz'$ is the mass of the element; and the height being small, we may suppose the element projected on the plane of xy. Hence the whole attraction parallel to x

$$= \rho \int_x^X \int_y^Y \int_0^H \frac{x'\,dx'\,dy'\,dz'}{\{x'^2+y'^2\}^{\frac{3}{2}}} = \rho H \int_x^X \int_y^Y \frac{x'\,dx'\,dy'}{\{x'^2+y'^2\}^{\frac{3}{2}}}$$

$$= \rho H \int_x^X \left\{ \frac{Y}{\sqrt{x'^2+Y^2}} - \frac{y}{\sqrt{x'^2+y^2}} \right\} \frac{dx'}{x'}$$

$$= \rho H \log_e \left\{ \frac{\sqrt{1+\frac{Y^2}{x^2}}+\frac{Y}{x}}{\sqrt{1+\frac{Y^2}{X^2}}+\frac{Y}{X}} \cdot \frac{\sqrt{1+\frac{y^2}{X^2}}+\frac{y}{X}}{\sqrt{1+\frac{y^2}{x^2}}+\frac{y}{x}} \right\}$$

To simplify the formula put

$$\frac{Y}{x} = \tan\theta_1, \quad \frac{y}{X} = \tan\theta_2, \quad \frac{Y}{X} = \tan\theta_3, \quad \frac{y}{x} = \tan\theta_4;$$

$$\therefore \sqrt{1 + \frac{Y^2}{x^2}} + \frac{Y}{x} = \frac{1 + \sin\theta_1}{\cos\theta_1} = \tan(45^\circ + \tfrac{1}{2}\theta_1),$$

and so of the rest. Hence, since 0·434 is the modulus of common logarithms,

$$\text{attraction} = \frac{\rho H}{0{\cdot}434}\Big\{\log\tan(45^\circ + \tfrac{1}{2}\theta_1) + \log\tan(45^\circ + \tfrac{1}{2}\theta_2) - \log\tan(45^\circ + \tfrac{1}{2}\theta_3) - \log\tan(45^\circ + \tfrac{1}{2}\theta_4)\Big\},$$

which gives a remarkably simple rule for finding the attraction parallel to x: that parallel to y can be found in like manner.

It is easy to show, that if the density be half the mean density of the earth, that is, about the same as granite, g be gravity, the radius of the earth = 20923713 feet, and H be expressed in feet, the coefficient above $= \dfrac{gH}{76127500}$.

This equals $gH\tan\left(\dfrac{1''}{569}\right)$. Hence, since the tangent of deflection of the plumb-line caused by the attraction equals, by the parallelogram of forces, the ratio of the attraction to gravity, and the angle is very small,

Deflection of plumb-line caused by the Tabular Mass parallel to the axis of x

$$= \frac{1''}{569} H\Big\{\log\tan(45^\circ + \tfrac{1}{2}\theta_1) + \log\tan(45^\circ + \tfrac{1}{2}\theta_2) - \log\tan(45^\circ + \tfrac{1}{2}\theta_3) - \log\tan(45^\circ + \tfrac{1}{2}\theta_4)\Big\}.$$

It is evident that the Tabular Mass may be partly below and partly above the plane of xy, so long as the height or depth is not so great that its square may not be neglected in comparison with the square of the distance from the attracted

point. In this case H is the sum of the height and depth, above and below the plane of xy.

Ex. 1. The co-ordinates to the nearest and furthest angles of a tabular block of rock measured from the attracted point are 3 and − 16, 40 and 30 miles, and the height of the mass from bottom to top is 628 feet. Show that the deflection of the plumb-line at the station taken as origin, and parallel to the shorter side of the parallelogram, = 3″·172.

Ex. 2. A table-land 1610 feet high, commencing at a distance of 20 miles from Takal K'hera, near the Great Arc of Meridian in India, runs 80 miles north, and 60 miles to the east and 60 to the west. Find the deviation of the plumb-line at that station. It is about 5″; so considerable as to have induced Sir G. Everest to abandon that place as a principal station.

56. In cases where the attracting mass is near, it is necessary to cut it up into prisms and calculate the effect of each separately and add the results. Examples of this are seen in the celebrated case of Schehallien, and more recently in the calculation of the deflection at Arthur's Seat, Edinburgh, by Sir H. James, Superintendent of the Ordnance Survey. See *Philosophical Transactions* for 1856, p. 591.

57. The irregular character of the surface of the Earth, consisting of mountain and valley and ocean, may in some instances have a sensible effect, by presenting an excess or deficiency of attracting matter, upon the position of the plumb-line, in such a way as to derange delicate survey operations. Hindostan affords a remarkable example of this, as the most extensive and the highest mountain-ground in the world lies to the north of that continent, and an unbroken expanse of ocean stretches south down to the south pole. Both these causes, by opposite effects, make the plumb-line hang somewhat north of the true vertical.

In the following Propositions a method is laid down for calculating the attraction of an irregular superficial stratum of the Earth's surface, and making it depend altogether upon the contour of the surface. The method pursued is this: A law of geometric dissection of the surface is discovered which

divides it into a number of four-sided spaces, such that if the height of the attracting mass were the same in them all, they would all attract the given station exactly to the same amount, whether far or near. In this case it would be necessary only to calculate for one space, then count the number of spaces in the country under consideration, and the final result is easily attained. The country being supposed irregular, the heights in the spaces will not be all alike. The principle, therefore, should be stated thus, that the attractions of the masses on the several compartments are in proportion to their mean heights. These mean heights are known by knowing the contour of the country.

PROP. *To discover a Law of Dissection of the surface of the earth into compartments, so that the attractions of the masses of matter standing on them, upon a given station in the horizontal direction, shall be exactly proportional to the mean heights of the masses, be they far or near.*

58. Suppose a number of great circles to be drawn from the station in question to the antipodes, making any angle β, each with the next, thus dividing the earth's surface (which we may in this calculation suppose to be a sphere, without incurring any sensible error) into a number of Lunes. Then, with the station as centre, describe on the surface a number of circles, at distances the law of which it is our object now to determine, dividing the whole into a number of four-sided compartments.

We will begin by calculating the attraction of a mass of matter, standing on one of these compartments, at a uniform height throughout, upon the station in a horizontal direction. Let α and $\alpha + \phi$ be the angular distances from the station of the two circles bounding this compartment; h the height of the mass; θ the angular distance along the surface of an elementary vertical prism of the mass; a the radius of the earth; ψ the angle which the plane of θ makes with the plane of the mid-line of the lune, and in which latter plane the resultant attraction evidently acts. The area of the base of the prism $= a^2 \sin\theta \, d\psi \, d\theta$.

Since the height of prism (h) is supposed very small, the

distances of its two extremities from the station may be taken to be the same, and $= 2a \sin \frac{1}{2}\theta$. Its attraction along the chord of θ

$$= \frac{\rho a^2 h \sin\theta \, d\theta \, d\psi}{4a^2 \sin^2 \frac{1}{2}\theta}.$$

Attraction along the tangent to $\theta = \frac{\rho h \sin\theta \, d\theta \, d\psi}{4 \sin^2 \frac{1}{2}\theta} \cos \frac{1}{2}\theta$;

$\therefore$ attraction along the tangent to the mid-line of the lune

$$= \frac{\rho h \cos^2 \frac{1}{2}\theta \, d\theta \, d\psi}{2 \sin \frac{1}{2}\theta} \cos\psi;$$

$\therefore$ attraction of the whole mass

$$= \frac{\rho h}{2} \int_{\alpha}^{\alpha+\phi} \int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \frac{\cos^2 \frac{1}{2}\theta}{\sin \frac{1}{2}\theta} d\theta \, . \cos\psi d\psi = \rho h \sin \tfrac{1}{2}\beta \int_{\alpha}^{\alpha+\phi} \frac{\cos^2 \frac{1}{2}\theta}{\sin \frac{1}{2}\theta} d\theta$$

$$= 2\rho h \sin \tfrac{1}{2}\beta \left\{ \log_e \frac{\tan \frac{1}{4}(\alpha+\phi)}{\tan \frac{1}{4}\alpha} + \cos \tfrac{1}{2}(\alpha+\phi) - \cos \tfrac{1}{2}\alpha \right\}$$

$$= 2\rho h \sin \tfrac{1}{2}\beta \left\{ \log_e \frac{\sin(\frac{1}{2}\alpha+\frac{1}{4}\phi) + \sin \frac{1}{4}\phi}{\sin(\frac{1}{2}\alpha+\frac{1}{4}\phi) - \sin \frac{1}{4}\phi} - 2\sin(\tfrac{1}{2}\alpha+\tfrac{1}{4}\phi) \sin \tfrac{1}{4}\phi \right\}$$

$$= 4\rho h \sin \tfrac{1}{2}\beta \sin \tfrac{1}{4}\phi \left\{ \frac{1}{\sin(\frac{1}{2}\alpha+\frac{1}{4}\phi)} - \sin(\tfrac{1}{2}\alpha+\tfrac{1}{4}\phi) \right\}$$

$$= 4\rho h \sin \tfrac{1}{2}\beta \frac{\sin \frac{1}{4}\phi \cos^2(\frac{1}{2}\alpha+\frac{1}{4}\phi)}{\sin(\frac{1}{2}\alpha+\frac{1}{4}\phi)}$$

neglecting only the cube and higher powers of $\sin \frac{1}{4}\phi$.

The law of dissection we shall choose will simplify this; for we are to assume such a relation between ϕ and α that the expression in ϕ may be constant, in order to make the attraction the same for all compartments in which h is the same or varying as h where the heights of the masses standing on the compartments are different. As the value of the constant to which we equal the function of α and ϕ is quite arbitrary,

we will assume it such that when α and ϕ are small, ϕ shall $= \frac{1}{10}\alpha$. In this case it $= \frac{\frac{1}{40}\alpha}{\frac{1}{2}\alpha + \frac{1}{40}\alpha} = \frac{1}{21}$.

Hence
$$\frac{\sin \frac{1}{4}\phi \cos^2 (\frac{1}{2}\alpha + \frac{1}{4}\phi)}{\sin (\frac{1}{2}\alpha + \frac{1}{4}\phi)} = \frac{1}{21} \quad \ldots\ldots\ldots\ldots (1),$$

defines the Law of Dissection.

The attraction of the mass standing on the compartment, in consequence,

$$= \frac{4}{21} \rho h \sin \tfrac{1}{2}\beta;$$

an exceedingly simple expression. We may obtain it in terms of gravity as follows. Let ρ the density be the same as that of the mountain Schehallien, viz. 2.75; the mean density, according to Mr Baily's repetition of the Cavendish experiment, being 5.66; g gravity; $a = 4000$ miles.

$$\text{Now } g = \frac{4\pi}{3} a \times \text{mean density} = \frac{4\pi}{3} a \frac{566}{275} \rho;$$

$\therefore$ attraction of mass on any compartment

$$= \frac{4}{21} \frac{3}{4\pi} \frac{275}{566} \frac{h}{a} \sin \tfrac{1}{2}\beta . \; g = 0.000005523 h \sin \tfrac{1}{2}\beta . g,$$

h being expressed in parts of a mile.

Since $0{\cdot}000005523 = \tan (1''.1392)$;

$\therefore$ deflection of the plumb-line caused by this attraction

$$= 1''.1392 h \sin \tfrac{1}{2}\beta \quad \ldots\ldots\ldots\ldots (2).$$

59. The method of using this formula is as follows. When the numerical values of the successive pairs of α and ϕ are determined by the solution of equation (1) giving the law of dissection, lay them and the lunes down on a map of the country the attraction of which is to be found. It will thus be covered with compartments. After examining the map,

write down the average heights of the masses standing on all the several compartments of any one lune; add them together, multiply the sum by $1''.1392 \sin \frac{1}{2}\beta$, and the equation (2) shows that we have the deflection caused by the mass on the whole lune in the vertical plane of its middle line. Multiply by the cosine and then the sine of the azimuth of that middle line, and we have the deflections in the meridian and the prime-vertical. The same being done for all the lunes, and the results added, we have the effects in meridian and prime-vertical produced by the whole country under consideration.

PROP. *To calculate the dimensions of the successive compartments from the law of dissection.*

60. For this purpose we should solve the equation of last Proposition, viz.

$$\frac{\sin \frac{1}{4}\phi \cos^2 (\frac{1}{2}\alpha + \frac{1}{4}\phi)}{\sin (\frac{1}{2}\alpha + \frac{1}{4}\phi)} = \frac{1}{21} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But this cannot be done. We must therefore approximate, which will equally well suit our purpose. In order to afford a test of the values we arrive at the equation may be written under the following form:

$$\log \sin \tfrac{1}{4}\phi^0 = 18.6777807 + \log \sin (\tfrac{1}{2}\alpha + \tfrac{1}{4}\phi) - 2 \log \cos (\tfrac{1}{2}\alpha + \tfrac{1}{4}\phi) \quad \ldots\ldots\ldots\ldots (3).$$

Equation (1) can be solved by expansion so long as α and ϕ are not too large.

It gives

$$\phi = \frac{4}{21}\left(\frac{\alpha}{2} + \frac{\phi}{4}\right)\left\{1 - \frac{1}{6}\left(\frac{\alpha}{2} + \frac{\phi}{4}\right)^2 + \left(\frac{\alpha}{2} + \frac{\phi}{4}\right)^2\right\}$$

$$= \frac{2}{21}\left(\alpha + \frac{\phi}{2}\right)\left\{1 + \frac{5}{6}\left(\frac{\alpha}{2} + \frac{\phi}{4}\right)^2\right\};$$

$$\therefore \frac{\alpha}{\phi} + \frac{1}{2} = \frac{21}{2}\left\{1 - \frac{5}{6}\left(\frac{\alpha}{2} + \frac{\phi}{4}\right)^2\right\},$$

$$\frac{\alpha}{\phi} + 1 = 11 - \frac{35}{16}(\alpha + \phi - \tfrac{1}{2}\phi)^2,$$

$$\frac{\phi}{\alpha + \phi} = \frac{1}{11}\left\{1 + \frac{35}{176}(\alpha + \phi - \tfrac{1}{2}\phi)^2\right\}$$

$$= \frac{1}{11}\left\{1 + \frac{35}{176}\left(\frac{21}{22}\right)^2(\alpha + \phi)^2\right\}$$

$$= \frac{1}{11}\{1 + 0.1812\,(\alpha + \phi)^2\}, \quad = \frac{1}{11}\{1 + 0.000055\,(\alpha + \phi)^2\} \ldots (4),$$

α and ϕ being expressed in degrees.

Let $\alpha_1\alpha_2\alpha_3 \ldots \phi_1\phi_2\phi_3 \ldots$ be the successive values of α and ϕ for the several compartments of a lune, beginning with the antipodes*. These are connected by the following relations :

$$\alpha_1 + \phi_1 = 180^\circ, \quad \alpha_2 + \phi_2 = \alpha_1, \quad \alpha_3 + \phi_3 = \alpha_2, \quad \&c. \ldots\ldots (5).$$

For the first, equation (1) gives

$$21 \sin^3 \tfrac{1}{4}\phi_1 = \cos \tfrac{1}{4}\phi_1, \quad \text{or} \quad \cot^3 \tfrac{1}{4}\phi_1 + \cot \tfrac{1}{4}\phi_1 - 21 = 0,$$

which gives

$$\cot \tfrac{1}{4}\phi_1 = 2.6379, \quad \text{or} \quad \phi_1 = 83^\circ 2', \quad \therefore \alpha_1 = 96^\circ 58'.$$

For the second, $\frac{1}{2}(\alpha_2 + \phi_2) = \frac{1}{2}\alpha_1 = 48^\circ 29'$. Putting this in equation (3) we must by trial find the value of ϕ_2 which satisfies it; and so of $\phi_3 \ldots$ This process brings out the series of values of ϕ_2 and α_2, ϕ_3 and α_3, &c. as far as the 22nd, gathered together in the following Table :

* In the last edition the compartments were counted from the station, not from the antipodes.

No. of Compart.	Values of ϕ	α	No. of Compart.	Values of ϕ	α	No. of Compart.	Values of ϕ	α
1	83° 2′	96°58′	18	1°29′	14°39′	35	0°18′	2°57′
2	15 12	81 46	19	1 21	13 18	36	0 16	2 41
3	10 54	70 52	20	1 13	12 5	37	0 15	2 26
4	8 34	62 18	21	1 6	10 59	38	0 13	2 13
5	7 3	55 15	22	1 0	9 59	39	0 12	2 1
6	5 58	49 17	23	0 54	9 5	40	0 11	1 50
7	5 8	44 9	24	0 50	8 15	41	0 10	1 40
8	4 29	39 40	25	0 45	7 30	42	0 9	1 31
9	3 56	35 44	26	0 41	6 59	43	0 8	1 23
10	3 29	32 15	27	0 38	6 21	44	0 8	1 15
11	3 6	29 9	28	0 35	5 46	45	0 7	1 8
12	2 47	26 22	29	0 31	5 15	46	0 6	1 2
13	2 30	23 52	30	0 29	4 46	47	0 6	0 56
14	2 15	21 37	31	0 26	4 20	48	0 5	0 51
15	2 1	19 36	32	0 24	3 56	49	0 5	0 46
16	1 49	17 47	33	0 21	3 35	50	0 4	0 42
17	1 39	16 8	34	0 20	3 15	&c.	&c.	&c.

After the 22nd each of the remaining values of ϕ is obtained by dividing the next preceding values of α by 11, as the small term in equation (4) then becomes insignificant, and the succeeding value of α is easily deduced by means of the formulæ (5). The Table may be carried on to any extent, the only restriction on its use being that the height of the mass on any compartment must not be so great relatively to its distance from the station that the square of the ratio cannot be neglected.

61. Cor. The relative effect of the same or an equal and similar mass, situated on different parts of the earth's surface, is easily obtained as follows.

As the effects of the compartments into which any lune is divided are all the same, the height of the mass standing on them being the same, the effect of a given mass standing on any area will vary inversely as the area of the particular compartment in which it is situated. Now if α and $\alpha + \phi$ be the distances of the nearer and further sides of any compartment, and β be the width of the lune, the area of the compartment $= \beta \{\cos \alpha - \cos (\alpha + \phi)\}$. Hence the relative attraction of the same mass in different situations will vary inversely as

$$\{\cos \alpha - \cos (\alpha + \phi)\}.$$

For example; the centre of the Island of Australia is about 36° and 63° from Singapoor and Calcutta; it stands therefore, with reference to those places, on the 9th and 4th compartments, reckoning from their antipodes, and the ratio of the horizontal attractions of the Island on those places

$$= \frac{\cos 62^\circ 18' - \cos 70^\circ 52'}{\cos 35^\circ 44' - \cos 39^\circ 40'} = \frac{0.46484 - 0.32777}{0.81174 - 0.76977}$$

$$= \frac{0.13707}{0.04197} = 3.266.$$

62. The formulæ above deduced may be applied to find the effect on the plumb-line of any mountain-region, or hollow (as in the case of the ocean), so long as the angle subtended at the station by any vertical line in it is such as to allow its square to be neglected.

Ex. 1. In the *Philosophical Transactions* for 1855 (p. 85) and 1859 (p. 770) the author has applied these principles to find the effect of the Himmalayas and the mountain-region beyond them on the plumb-line in India, and has found that the meridian deflection caused in the northern station of the Great Arc of Meridian (lat. 29° 30′ 48″, and long. 77° 42′) is nearly 28″, as far as the data regarding the contour of the mass have been ascertained; and that the astronomical amplitudes between that and the next principal station (lat. 24° 7′ 11″), and between that and the third (lat. 18° 3′ 15″), are diminished by the quantities 15″.9 and 5″.3. He has also shown that the meridian deflection at points between the first and third stations varies very nearly inversely as the distance from a point in the meridian in latitude 33° 30′.

General Chodzko states that at Tiflis, Douchet, Wladikawkas, Alexandrowskaja, and Mosdok, which are severally 70, 35, 35, 55, 70 miles from the central line of the Caucasus, the deflections are (taking that at Tiflis to be zero) 25″.1, − 28″.6, − 12″.0, − 5″.6 North. (See *Monthly Notices of Astron. Soc.* April, 1862.)

Ex. 2. The effect of the deficiency of matter in the Ocean south of Hindostan down to the south pole is also calculated (*Phil. Trans.* 1859, p. 790) by the author, upon an assumed

but not improbable law of the depth, and found to produce a meridian deflection northwards at the three stations of the Indian Arc of 6″, 9″, 10″.5 respectively; and 19″.7 at Cape Comorin. The deflections at Karachi, and a point half way between Cape Comorin and Karachi, arising from this cause, are shown to be 10″ and 13″.8.

It is not difficult to show from the last three, that the horizontal attraction northwards, at points along the west coast of India, arising from deficiency of matter in the ocean, may be approximately represented by the formula

$$(0.000095556839 - 0.000002836162\lambda + 0.000000004072\lambda^2)\,g,$$

in which λ is the difference of latitude of the station and Karachi, expressed in degrees and parts of a degree. (*Phil. Trans.* 1859, p. 793.)

Ex. 3. The formulæ may be applied also to obtain the attraction of thin sections of the earth's surface of a *regular* form which the Integral Calculus does not enable us to calculate. The following is an example which the reader may work out: the result is here given because it will be used in the last chapter of this treatise.

The horizontal attraction of a slender hemi-spheroidal meniscus of matter at the earth's surface on points 90°, 120°, 135°, 150°, 180° from the pole of the meniscus is

$$0{\cdot}1202\frac{h}{a}g,\quad 0{\cdot}0412\frac{h}{a}g,\quad 0{\cdot}0236\frac{h}{a}g,\quad 0{\cdot}0138\frac{h}{a}g,\quad 0;$$

h = greatest thickness of the meniscus, a = radius of the earth. If θ be the distance of any point in the further hemisphere from the pole of the meniscus the above quantities lead to the following formula. Horizontal attraction

$$= (0{\cdot}1446\sin\theta + 0{\cdot}0958\sin 2\theta + 0{\cdot}0244\sin 3\theta)\frac{h}{a}g \ldots (1),$$

which may be taken as representing generally the attraction at any point of the hemisphere of the meniscus.

By means of Art. 14 it may be shown, that the attraction of the difference of two spheroids of different small ellipticity

having the same equator, i.e. of a meniscus in the other hemisphere taken together with the meniscus we have been considering,

$$= 0{\cdot}6 \sin 2\phi \frac{h}{a} g,$$

ϕ being the distance from the nearest pole. Hence if we take the difference of these we have the attraction of a thin hemi-spheroidal meniscus on a point on its own surface: the formula becomes, attending to the directions of the attraction, Horizontal attraction

$$= (0{\cdot}1446 \sin \phi - 0{\cdot}6958 \sin 2\phi + 0{\cdot}0244 \sin 3\phi) \frac{h}{a} g \ldots (2).$$

$\phi = 180^\circ - \theta$, so that θ and ϕ in (1) and (2) are each measured from the pole of the attracting meniscus, and in each case the attraction is reckoned positive towards the pole of the attracting meniscus.

Ex. 4. A somewhat simpler example for the reader to work out is this, To find the tangential attraction of a hemi-spherical shell of small uniform thickness upon any point in the surface of the whole sphere.

If the calculation be first made for points, as in the last example*, 91°, 120°, 135°, 150°, 180° from the pole of the shell, the results will be

$$1{\cdot}3750 \frac{h}{a} g, \quad 0{\cdot}1916 \frac{h}{a} g, \quad 0{\cdot}1128 \frac{h}{a} g, \quad 0{\cdot}0852 \frac{h}{a} g, \quad 0;$$

and the following formula will approximately embrace other points in the hemisphere opposite to the hemi-spherical shell: Horizontal attraction

$$= (2{\cdot}2054 \sin \theta + 1{\cdot}9842 \sin 2\theta + 0{\cdot}7606 \sin 3\theta) \frac{h}{a} g \ldots (3).$$

As the tangential attraction of a whole spherical shell on any point is zero, it follows that the tangential attraction of

* The first point is here taken 91° and not 90° (that is, 1° or about 70 miles from the edge of the shell) because otherwise the square of the ratio of the height of the mass on the nearest compartments to the distance from the point attracted could not be neglected. See end of Art. 60.

a hemi-spherical shell on any point on its own surface will equal the above with its sign changed: or if ϕ be the angle from the pole of the shell it will be

$$(2{\cdot}2054 \sin \phi - 1{\cdot}9842 \sin 2\phi + 0{\cdot}7606 \sin 3\phi) \frac{h}{a} g \ldots (4),$$

θ and ϕ being reckoned in each case from the pole of the shell. In each case the attraction is reckoned positive towards the pole of the attracting meniscus.

Ex. 5. Suppose we take a hemi-spheroidal meniscus of thickness h at its edge, and no thickness at the pole. The attraction of this will be found by subtracting the results of Ex. 3 from those of Ex. 4: they give

$$(2{\cdot}0608 \sin \theta + 1{\cdot}8884 \sin 2\theta + 0{\cdot}7362 \sin 3\theta) \frac{h}{a} g \ldots (5),$$

and $$(2{\cdot}0608 \sin \phi - 1{\cdot}2884 \sin 2\phi + 0{\cdot}7362 \sin 3\phi) \frac{h}{a} g \ldots (6).$$

63. It is possible that, the superabundant matter in mountain-regions having been heaved up from below, there may be a deficiency of matter below the mountains which would under certain circumstances have the tendency of counteracting their effect on the plumb-line. This Mr Airy has suggested in a Paper in the *Philosophical Transactions* of 1855, on the hypothesis that the deficiency is immediately below the mountains close to their mass. Upon the supposition that the mountains may have drawn their mass from the regions below through a considerable depth, by an extensive and small expansion of the matter in those lower regions, the author has calculated the modifying effect on the plumb-line in the *Philosophical Transactions* for 1858—9. This has brought to light the fact, that a trifling deviation in the density from that required for fluid-equilibrium, if it prevail through extensive tracts, may have a sensible effect upon the plumb-line. This has been recently verified by the observations and calculations of Professor Schmeizer, who has shown that within a distance of 19 miles the plumb-line varies by 16″ near Moscow without any apparent cause, and that it varies in such a way as to indicate a deficiency of matter

below. See *Monthly Notices of Ast. Soc. Ap.* 1862. The following Proposition, with which we shall close this Chapter, will show that this is possible. These questions, in themselves interesting as problems in Attraction, become still more so, as we shall see, in the determination of the Figure of the Earth.

PROP. *To find the effect on the plumb-line of a slight but wide-spread deviation in density in the interior of the earth, either in excess or defect, from that required by the laws of fluid-equilibrium.*

64. Suppose a four-sided space drawn upon the surface of the earth, bounded on two sides by great circles passing through the station where the plumb-line is and making an angle β with each other, the other two sides being parts of circles of which the station is the centre; let ϕ be the angular distance of these two circles measured along the surface, and θ the distance of the middle of ϕ from the station. We shall take $\phi = 2^\circ\ 52'\ 40''$ (= 200 miles), and $\beta = 30^\circ$, and shall find how small θ may be that a mass of small uniform height covering the space should attract the station as if it were collected into the middle point of ϕ. The area of the space

$$= a^2\beta\{\cos(\theta - \tfrac{1}{2}\phi) - \cos(\theta + \tfrac{1}{2}\phi)\} = 2a^2\beta \sin \tfrac{1}{2}\phi \sin\theta,$$

and the chord between the mid-point and the station being $2a \sin \frac{1}{2}\theta$, the attraction of the mass collected at the mid-point and resolved along the tangent

$$= \frac{2\rho h a^2 \beta \sin \frac{1}{2}\phi \sin\theta}{4a^2 \sin^2 \frac{1}{2}\theta} \cos \tfrac{1}{2}\theta = \rho h \beta \sin \tfrac{1}{2}\phi \frac{\cos^2 \frac{1}{2}\theta}{\sin \frac{1}{2}\theta}.$$

But by Art. 58 the attraction of the mass

$$= 2\rho h \sin \tfrac{1}{2}\beta \left(\log_e \frac{\sin \frac{1}{2}\theta + \sin \frac{1}{4}\phi}{\sin \frac{1}{2}\theta - \sin \frac{1}{4}\phi} - 2 \sin \tfrac{1}{2}\theta \sin \tfrac{1}{4}\phi \right)$$

$$= 4\rho h \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\theta \sin \tfrac{1}{4}\phi \left(\frac{1}{\sin^2 \frac{1}{2}\theta} + \frac{\sin^2 \frac{1}{4}\phi}{3 \sin^4 \frac{1}{2}\theta} + \ldots - 1 \right)$$

$$= \rho h \beta \sin \tfrac{1}{2}\phi \left(1 - \frac{\beta^2}{24} + \frac{\phi^2}{32} + \frac{4 \sin^2 \frac{1}{4}\phi}{3 \sin^2 \theta} \right) \frac{\cos^2 \frac{1}{2}\theta}{\sin \frac{1}{2}\theta}.$$

This coincides with the previous expression if

$$\frac{4\sin^2\frac{1}{4}\phi}{3\sin^2\theta} - \frac{\beta^2}{24} + \frac{\phi^2}{32} = \text{a very small quantity} = \frac{1}{100} \text{ suppose.}$$

$$\text{Put } \beta = 30^\circ = \pi \div 6,\ \phi = 2^\circ\,52'\,40'' = 0{\cdot}016\pi;$$

$$\therefore\ 4\sin^2\tfrac{1}{4}\phi = 3\sin^2\theta\,(0{\cdot}01 + 0{\cdot}012) = 0{\cdot}066\sin^2\theta;$$

$$\therefore\ \sin\theta = 8\sin\tfrac{1}{4}\phi \text{ nearly};\ \ \therefore\ \theta = 2\phi = 400 \text{ miles.}$$

Hence the centre of the space may be as near as 400 miles to the station, and yet the whole mass be supposed to be collected into its centre. The area $= a^2\beta\phi\sin\theta = 2a^2\beta\phi^2 = (a\phi)^2$ very nearly $= (200)^2$ miles, or the space is equal to a square of 200 miles each way.

65. Now suppose the height of the matter on this space to be 1 mile, and suppose every small vertical prism of it to be distributed uniformly downwards into a slender prism to a depth d. Thus the whole superficial mass 1 mile thick will be distributed through a depth d, and form an attenuated mass the density of which is one d^{th} part of that of the superficial rock. As the mass at the surface may be collected into its middle point, much more may that in any horizontal section of this attenuated mass, because the section is further from the station than the space at the surface. Hence the whole attenuated mass will attract the station as if it were collected uniformly into one vertical prism drawn down from the central point of the surface to the depth d. Let u and v be the distances of the extremities of this prism from the station:—

Therefore attraction on the station along u

$$= \frac{\text{mass}}{uv} = \frac{\rho\times 200^2}{uv} \text{ (Art. 48)} = \frac{1.2}{uv}g.$$

This will also be approximately the *horizontal* attraction for all distances not exceeding 30° from the station.

Hence deflection of the plumb-line

$$= \frac{1.2}{uv} \text{ in arc} = \frac{1000000''}{4uv}.$$

Ex. We may give any values to u and v so long as u is not less than 400 miles. We shall take $u = 400$, 600, 800, 1000 miles successively. The calculation will be facilitated by using a table of tangents and secants, observing that $u \div d$ is the tangent of the angle of which $v \div d$ is the secant. Hence the following Table:

Depth in miles.	Distance of the mid-point of the space from the station, measured along the chord, in miles; viz.							
	$u = 400$		600		800		1000	
	$\frac{u}{d}$'	$\frac{v}{d}$	$\frac{u}{d}$'	$\frac{v}{d}$	$\frac{u}{d}$'	$\frac{v}{d}$	$\frac{u}{d}$'	$\frac{v}{d}$
100	4.00	4.12	6.00	6.08	8.00	8.06	10.00	10.05
200	2.00	2.24	3.00	3.16	4.00	4.12	5.00	5.10
300	1.33	1.66	2.00	2.24	2.67	2.85	3.33	3.48
900	0.44	1.09	0.67	1.20	0.89	1.34	1.11	1.49
1000	0.40	1.08	0.60	1.17	0.80	1.28	1.00	1.41

This Table enables us, with the formula above, to tabulate the deflections as follows:

Deflections, caused by the mass distributed downwards through a depth of 100 miles.	Distance of mid-point from the station, along the chord, in miles.			
	400	600	800	1000
Ditto 200 „	1″·51	0″.69	0″.39	0″.25
Ditto 300 „	1 .40	0 .66	0 .38	0 .25
Ditto 900 „	1 .25	0 .62	0 .36	0 .24
Ditto 1000 „	0 .64	0 .38	0 .26	0 .18
	0 .58	0 .36	0 .24	0 .18

The densities of the masses distributed through the depths 100, 200, 300, 900, 1000 miles are severally inversely proportional to those numbers. Hence by multiplying the lines of numbers in this table successively by 1, 2, 3, 9, 10 we shall have the deflections of masses having the same volumes as

before, but all of the same density, viz. 1-100th part of that of superficial rock. The numbers then are

1".51	0".69	0".39	0".25
2.80	1.32	0.76	0.50
3.75	1.86	1.08	0.72
5.76	3.42	2.34	1.62
5.80	3.60	2.40	1.80

Subtract each line from the line below (except the 3rd line) and we obtain the following

Deflections caused by a semi-cubic mass, 200 miles in each horizontal side and 100 miles deep, density = 1-100th of the density of the surface, and depth of the centre = 50 miles	Distance of the mid-point from the station, along the chord, in miles. 400	600	800	1000
	1".51	0".69	0".39	0".25
Ditto - - 150 ,,	1 .29	0 .63	0 .37	0 .25
Ditto - - 250 ,,	0 .95	0 .54	0 .32	0 .22
Ditto - - 950 ,,	0 .04	0 .18	0 .06	0 .18

The horizontal dimensions of the spaces will be somewhat contracted in passing downwards owing to the convergence of the sides towards the centre of the earth: but the densities from the distribution downwards in slender prisms of uniform mass will increase in a corresponding degree: and the masses of the spaces will be all the same.

The last change we shall make is this. We shall increase the density of the semi-cubic space as its depth increases, so as to make it 1-100th part, not of the superficial density as at present, but of the density of the earth's mass at the centre of the space.

If D be the density of the surface, a the earth's radius, the usually received law of density of the interior is

$$\text{density at depth } d = \frac{2aD}{a-d} \sin\left(\frac{5\pi}{6}\,\frac{a-d}{a}\right),$$

when $d = 50, 150, 250, 950$ miles, this gives the ratio of the density at these depths to the superficial density = 1.17, 1.21, 1.35, 2.39. Multiply the deflections last found by these numbers, and we have finally

Deflections caused by an excess or defect of matter prevailing through a semi-cubic space 200 miles in each horizontal side and 100 miles deep, the density of the excess or defect being 1-100th of the earth's density at the centre of the semi-cubic space, when that centre is	Distance of the mid-point of the semi-cubic space from the station, measured along the chord, in miles.			
	400	600	800	1000
50 miles deep	1″.77	0″.81	0″.46	0″.29
Ditto - - 150 „	1 .56	0 .76	0 .45	0 .30
Ditto - - 250 „	1 .28	0 .73	0 .43	0 .30
Ditto - - 950 „	0 .10	0 .43	0 .14	0 .43

The defect or excess in density which we have taken, viz. 1-100th, might have been chosen larger, and the deflections proportionably increased. For there are many kinds of rock, as granite, which differ so in density in the different specimens that the difference between the extremes is greater even than 1-10th of the mean. And if this difference exists at the surface, it does not seem to be improper to suppose that great variations may exist also below, from the effect of the cooling down and solidifying of the crust, even much greater than 1-100th.

66. We have taken a semi-cubic space as our example: but the same result is true of a space of the same volume and of any form so long as its dimensions in one direction are not much larger than in another. This follows from Art. 64.

FIGURE OF THE EARTH.

INTRODUCTION.

67. It is easy to show in a general way, that the earth is a more or less spherical mass.

The globular form is seen in the shadow which the earth casts on the moon in eclipses in a variety of positions. The comparison of the distance at which ships at sea lose sight of each other's decks, with the height of the decks from the water, shows all over the world that the sea is of a globular form; and an approximation to the diameter of the globe is thus obtained by simple geometry. The distance of the horizon at sea as seen from cliffs and hills, the height of which is known, leads to the same result. The distance north and south between two places, measured, for instance, by a perambulator, is always found to be nearly in proportion to the difference of latitude; this could not be the case, if the curve of the meridian were not nearly circular.

After it was known that the earth is of a globular form, Newton was the first who demonstrated that it is not a perfect sphere. From theoretical considerations and also from the discovery that a pendulum moves slower at the equator than in higher latitudes, he arrived at the conclusion that its form is that of an oblate spheroid—the form being derived from rotation in a fluid state. This subject we propose now to consider. We shall in the first Chapter treat it on the hypothesis that the Earth was a fluid mass when it assumed its present general form. The calculation is one of great difficulty, and would indeed be impracticable did we not know that the figure differs but little from a sphere. In the second Chapter we shall show how the actual form is found by geodesy.

CHAPTER I.

THE FIGURE OF THE EARTH CONSIDERED AS A FLUID MASS.

§ 1. *The Earth considered to be a fluid homogeneous mass.*

As a first approximation we shall inquire whether a homogeneous fluid mass revolving about a fixed axis can be made to maintain a spheroidal form according to the laws of fluid pressure.

PROP. *A homogeneous mass of fluid in the form of a spheroid revolves with a uniform velocity about an axis: required to determine whether the equilibrium of the surface left free is possible.*

68. Let a and b be the semi-axes of the spheroid referred to three axes of rectangular co-ordinates, b being that about which it revolves: also let $b^2 = a^2(1-e^2)$. The forces which act upon the particle (xyz) are the centrifugal force and the attraction of the spheroid parallel to the axes: these latter are given in Art. 12, and are

$$\frac{2\pi\rho}{e^3}\{\sqrt{1-e^2}\sin^{-1}e - e(1-e^2)\}x,$$

$$\frac{2\pi\rho}{e^3}\{\sqrt{1-e^2}\sin^{-1}e - e(1-e^2)\}y,$$

$$\frac{4\pi\rho}{e^3}\{e - \sqrt{1-e^2}\sin^{-1}e\}z.$$

Let these be represented by Ax, By, Cz. Let w be the angular velocity of the rotation, then $w^2\sqrt{x^2+y^2}$ is the cen-

trifugal force of the particle (xyz), and the resolved parts of it parallel to the axes of x, y, z are $w^2x, w^2y, 0$. Hence X, Y, Z, the forces acting on (xyz) parallel to the axes, are

$$X = -(A - w^2)\,x, \quad Y = -(B - w^2)\,y, \quad Z = -Cz.$$

These make $Xdx + Ydy + Zdz$ a perfect differential, and therefore so far the equilibrium is possible.

The equation of fluid equilibrium gives

$$\frac{1}{\rho}dp = Xdx + Ydy + Zdz$$
$$= -(A - w^2)(xdx + ydy) - Czdz;$$

$$\therefore \frac{2p}{\rho} = \text{constant} - (A - w^2)(x^2 + y^2) - Cz^2.$$

At the surface $p = 0$, and therefore

$$\frac{A - w^2}{C}(x^2 + y^2) + z^2 = \text{const.}$$

is the equation to the surface; and this is a spheroid, and therefore the equilibrium is possible, the form of the spheroid being properly assumed. The eccentricity is given by the condition

$$1 - e^2 = \frac{b^2}{a^2} = \frac{A - w^2}{C},$$

$$\text{or } \frac{w^2}{2\pi\rho} = \frac{\sqrt{1 - e^2}}{e^3}\sin^{-1}e - 3\,\frac{1 - e^2}{e^2} + \frac{2}{e^3}(1 - e^2)^{\frac{3}{2}}\sin^{-1}e;$$

$$\therefore \frac{w^2}{2\pi\rho} + 3\,\frac{1 - e^2}{e^2} - \frac{(3 - 2e^2)\sqrt{1 - e^2}}{e^3}\sin^{-1}e = 0.$$

Now observation shows that $\frac{1}{289}$ = the ratio of the centrifugal force at the equator to gravity at the equator. Hence

$$\frac{1}{289} = \frac{w^2 a}{\frac{4}{3}\pi\rho a - w^2 a}; \quad \therefore \frac{w^2}{2\pi\rho} = \frac{1}{435}.$$

By expanding in powers of e and neglecting powers higher

than the second, because we know that the earth is nearly spherical, we have

$$\sin^{-1} e = e + \frac{1}{2}\frac{e^3}{3} + \frac{1.3}{2.4}\frac{e^5}{5} + \dots$$

$$\sqrt{1-e^2} = 1 - \frac{1}{2}e^2 - \frac{1.1}{2.4}e^4;$$

$$\therefore \frac{1}{435} = \left(\frac{3}{e^2} - 2\right)\left(1 - \frac{1}{2}e^2 - \frac{1}{8}e^4\right)\left(1 + \frac{1}{6}e^2 + \frac{3}{40}e^4\right) - \frac{3}{e^2} + 3$$

$$= \left(\frac{3}{e^2} - 2\right)\left(1 - \frac{1}{3}e^2 - \frac{2}{15}e^4\right) - \frac{3}{e^2} + 3 = \frac{4e^2}{15};$$

$$\therefore e^2 = \frac{1}{116}.$$

If ϵ be the ellipticity, then

$$\epsilon = \frac{a-b}{a} = 1 - \sqrt{1-e^2} = \frac{1}{2}e^2 = \frac{1}{232}.$$

This result is so much greater than that obtained by other methods, as we shall see, that it decides against our considering the earth's mass to be homogeneous. Indeed it is *à priori* highly improbable that the mass should be homogeneous, since the pressure must increase in passing towards the centre and the matter be in consequence compressed.

69. Another value of e, nearly $= 1$, satisfies the equation. But this does not give the figure of any of the heavenly bodies, since none of them are very elliptical.

Since there are two values of e which satisfy the equation, it might be supposed that the equilibrium of the mass under one of these forms would be unstable, and, upon any derangement taking place, the fluid would pass to the other as a stable form. But Laplace has shown (*Méc. Céles.* Liv. III. § 21) that for a given primitive impulse there is but one form. In fact it is easily seen that for a given value of w, the angular velocity, the vis viva of two equal masses, so different in their form as to have e small and nearly equal unity, must be

very different, and that therefore the mass cannot pass from one form to the other without a new impulse from without being given to its parts.

70. The relation between w and e in Art. 68, shows that as w alters e alters, and vice versâ. By putting $\frac{dw}{de} = 0$, we find the greatest value of w which is consistent with equilibrium. This after some long numerical calculations gives

$$e = \frac{17197}{27197}, \text{ and time of rotation} = 0{\cdot}1009 \text{ day.}$$

71. Before proceeding to calculate the ellipticity on the hypothesis of the earth's mass being heterogeneous we will take the following extreme case. The density increases as we pass down towards the centre. Suppose that at the centre it is infinitely greater than elsewhere: that is, suppose the whole force resides in the centre. The case of nature must lie between this hypothesis and that of the earth's being homogeneous.

PROP. *To calculate the ellipticity of a mass of fluid revolving about a fixed axis and attracted by a force residing wholly in the centre of the fluid and varying inversely as the square of the distance.*

72. Let M be the mass of the fluid; the other quantities as before;

$$\therefore X = -\frac{Mx}{r^3} + w^2x, \quad Y = -\frac{My}{r^3} + w^2y, \quad Z = -\frac{Mz}{r^3}.$$

Then the equation $Xdx + Ydy + Zdz = 0$ becomes

$$\frac{M}{r^3}(xdx + ydy + zdz) - w^2(xdx + ydy) = 0;$$

$$\therefore \frac{M}{r^2}dr - \frac{w^2}{2}d(x^2 + y^2) = 0;$$

$$\therefore \frac{M}{r} + \frac{w^2}{2}(x^2 + y^2) = \text{constant} = C.$$

As in Art. 68,

$$\frac{1}{289} = \frac{w^2 a}{\frac{M}{a^2} - w^2 a}, \quad \therefore \frac{M}{w^2 a^3} = 290;$$

$$\therefore \frac{1}{r} \text{ or } \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{C}{M} - \frac{1}{580}\frac{x^2+y^2}{a^3}.$$

By reversing this, squaring, expanding, and neglecting the square of $\frac{1}{580}$, this is seen to be the equation to a spheroid.

When $x=0$ and $y=0$, then $z=b$; when $z=0$, $x^2+y^2=a^2$;

$$\therefore \frac{1}{b} = \frac{C}{M}, \quad \frac{1}{a} = \frac{C}{M} - \frac{1}{580}\frac{1}{a}, \quad \frac{b}{a} = \frac{580}{581};$$

$$\therefore \epsilon = \frac{1}{581}.$$

This value of ϵ is too small (as we might have expected), as $\frac{1}{232}$ is too large, to agree with the form deduced by actual measurement by geodesy.

§ 2. *The Earth considered to be a fluid heterogeneous mass.*

73. From what has gone before it is clear that the earth's mass is not of uniform density throughout. This result indeed we might have anticipated. We shall now enter upon the more general theory of considering the mass to be heterogeneous in its density.

PROP. *To prove that if the mass of the earth is heterogeneous it must lie in strata nearly spherical about the earth's centre.*

74. The truth of this Proposition rests upon these two facts, which are obtained from observation: (1) That the external surface is nearly spherical; (2) That the force of gravity tends nearly towards the earth's centre. Let r, θ, ω be the co-ordi-

nates from the centre of any point of the surface, ($\cos\theta = \mu$), and let $r = a + \alpha \,.\, u$, where a is the mean radius, u a function of μ and ω, and α a small constant, the square of which may be neglected because the surface is nearly spherical, $r'\ \theta'\ \omega'$ co-ordinates to any point in the interior of the mass, ($\cos\theta' = \mu'$), ρ' the density at this point.

Then (Art. 19) the potential of the whole mass at the point on the surface is

$$V = \int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{r'} \frac{\rho' r'^2 d\mu' d\omega' dr'}{\sqrt{r^2 + r'^2 - 2rr'p}},$$

where $p = \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega - \omega')$. By expansion this becomes

$$V = \int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{r'} \rho' \left\{ \frac{r'^2}{r} P_0 + \ldots + \frac{r'^{i+2}}{r^{i+1}} P_i + \ldots \right\} d\mu' d\omega' dr',$$

where $P_0 \ldots P_i \ldots$ are Laplace's Coefficients.

Put $\rho' = R' + \beta \,.\, U'$, where R' is a function of r' only, independent of μ' and ω', and U' is a function of all three $r'\ \mu'\ \omega'$, β a constant. We have to prove that β is a small quantity of the order of α.

Also suppose

$$\int_0^{r'} r'^{i+2}\rho' dr' = \phi(r') + \beta \,.\, \psi(r', \mu', \omega') = \phi(a + \alpha u') + \beta\psi(a + \alpha u', \mu', \omega'),$$

$$= A + \alpha B\,(u'_0 + u'_1 + \ldots\ldots + u'_i + \ldots\ldots) + \beta\,(\psi'_0 + \psi'_1 + \ldots\ldots \psi'_i + \ldots\ldots),$$

these being series of Laplace's Functions. Then remembering their property proved in Art. 27,

$$V = \int_{-1}^{1}\int_{0}^{2\pi} \left\{ C\frac{P_0}{r} + \ldots + \frac{P_i}{r^{i+1}}\,(\alpha B u'_i + \beta\psi'_i) \ldots \right\} d\mu' d\omega'$$

$$= \frac{4\pi}{r}\,C + \ldots + \frac{4\pi}{(2i+1)\,r^{i+1}}\,(\alpha B u_i + \beta\psi_i) + \ldots,$$

by the property proved in Art. 32: C is a constant.

Now since the force of gravity acts very nearly towards the centre of the earth, the quantities (see Art. 20), $\frac{dV}{d\mu}$ and $\frac{dV}{d\omega}$, which depend upon the parts of gravity at right angles to r, must both be very small. Hence

$$\alpha B \frac{du_i}{d\mu} + \beta \frac{d\psi_i}{d\mu}, \quad \alpha B \frac{du_i}{d\omega} + \beta \frac{d\psi_i}{d\mu}$$

must be small: and this must be the case for all values of μ and ω, that is for every spot on the earth's surface. This cannot be the case unless β be small as well as α.

Hence the terms in ρ' which depend upon μ' and ω' are very small. From this it follows that ρ' may be regarded as a function of $r' + \alpha . \nu'$ where ν' is some function of r', μ', ω'.

$$\therefore \; r' + \alpha . \nu' = \text{constant}$$

will be the general equation to layers of equal density. This is evidently the equation to a surface nearly spherical around the origin of r'. Hence the mass lies in strata nearly spherical about the earth's centre.

PROP. *To find the equation of equilibrium of a heterogeneous mass of fluid consisting of strata each nearly spherical, and revolving about a fixed axis passing through the centre of gravity with a uniform angular velocity.*

75. Let XYZ be the sums of the resolved parts of all the forces which act upon any particle (xyz) of the fluid, parallel to the axes of co-ordinates, ρ' the density at that point, p the pressure. Then the equation of fluid equilibrium is

$$\frac{dp}{\rho'} = Xdx + Ydy + Zdz.$$

At the surface, and also throughout any internal stratum of equal pressure and therefore of equal density, in passing from point to point $dp = 0$.

Hence $$Xdx + Ydy + Zdz = 0$$

is the differential equation to the exterior surface and to the surfaces of all the internal strata; the particular value assigned

to the constant after integration determining to which surface the integral belongs.

The following property belongs to all these surfaces. If ds be the element of any curve drawn on the surface through (xyz), and R be the resultant of XYZ; then the equation may be written

$$\frac{X}{R}\frac{dx}{ds}+\frac{Y}{R}\frac{dy}{ds}+\frac{Z}{R}\frac{dz}{ds}=0.$$

The first side of this is the cosine of the angle between the resultant and the line ds, and as it equals zero it shows that the resultant force is at right angles to any line in the surface, and therefore to the surface itself at the point (xyz).

The equilibrium will be the same if we suppose the rotatory motion not to exist, but apply to each particle a force equal to the centrifugal force caused by the rotation. The forces then acting on the fluid will be the centrifugal force and the mutual attraction of the parts of the fluid. Let V be the potential (Art. 18) for this mass, then

$$-\frac{dV}{dx},\ -\frac{dV}{dy},\ -\frac{dV}{dz}$$

are the attractions parallel to the three axes tending towards the origin of co-ordinates. Let w be the angular velocity of rotation about the axis of z, taken as the fixed axis; $w^2.x$ and $w^2.y$ will be the centrifugal force at the point (xyz). Then

$$\frac{dp}{\rho'}=\left(\frac{dV}{dx}+w^2x\right)dx+\left(\frac{dV}{dy}+w^2y\right)dy+\frac{dV}{dz}dz,$$

$$\therefore \text{ constant or } \int\frac{dp}{\rho'}=V+\frac{w^2}{2}(x^2+y^2)$$

is the equation to the surface and the strata.

Let r be the distance of the point (xyz) from the origin, and θ the angle r makes with the axis of z, and $\cos\theta=\mu$: then $x^2+y^2=r^2\sin^2\theta=(1-\mu^2)r^2$. Also let m be the ratio of the centrifugal force at the equator to gravity at the

equator $\left(\text{or } \frac{1}{289}\right)$; let a' be the mean radius of the stratum through (xyz); a the radius of the equator; then

$$m = w^2 \mathrm{a} \div \frac{M}{\mathrm{a}^2} = \frac{w^2 \mathrm{a}^3}{M},$$

$$\text{and } M = 4\pi \int_0^{\mathrm{a}} \rho' a'^2 da' = \frac{4}{3}\pi\phi(\mathrm{a}) \text{ suppose,}$$

the strata being considered spherical because of the smallness of the numerator in the value of m;

$$\therefore\ m = \frac{3w^2\mathrm{a}^3}{4\pi\phi\,(\mathrm{a})},\ \therefore\ w^2 = \frac{4\pi}{3}\, m\, \frac{\phi\,(\mathrm{a})}{\mathrm{a}^3},$$

and the equation becomes

$$\text{constant or } \int \frac{dp}{\rho'} = V + \frac{2\pi}{3}\, m\, \frac{\phi\,(\mathrm{a})}{\mathrm{a}^3}\,(1-\mu^2)r^2$$

$$= V + \frac{4\pi}{9}\, m\, \frac{\phi\,(\mathrm{a})}{\mathrm{a}^3}\, r^2 + \frac{2\pi}{3}\, m\, \frac{\phi\,(\mathrm{a})}{\mathrm{a}^3}\left(\frac{1}{3} - \mu^2\right) r^2,$$

this arrangement being made, because the second and third terms as they now stand, are Laplace's Functions of the order 0 and 2. (See Art. 39, Ex. 1.)

By Art. 46, we have

$$V = \frac{4\pi}{r}\int_0^a \rho' \left\{a'^2 + \frac{d}{da'}\left(\frac{a'^4}{3r}\, Y_1' + \ldots + \frac{a'^{i+3}}{(2i+1)\, r^i}\, Y_i' + \ldots\right)\right\} da'$$

$$+ 4\pi \int_a^{\mathrm{a}} \rho' \left\{a' + \frac{d}{da'}\left(\frac{a'r}{3}\, Y_1' + \ldots + \frac{r^i}{(2i+1)\, a'^{i-2}}\, Y_i' + \ldots\right)\right\} da'.$$

In this put

$$r = a\,(1 + Y_1 + \ldots\ Y_i + \ldots) \text{ and } \int_0^a \rho' a'^2 da' = \tfrac{1}{3}\phi\,(a),$$

as before. Then substitute this value of V in the equation to the strata and equate terms of the order i. (See Art. 35.)

The constant parts give

$$\int\frac{dp}{\rho'}=\frac{4\pi}{3}\frac{\phi(a)}{a}+4\pi\int_a^a\rho' a'da'+\frac{4\pi}{9}ma^2\frac{\phi(a)}{a^3} \quad \ldots\ldots (1)$$

and the terms of the order i give

$$\frac{\phi(a)}{3a}Y_i-\frac{1}{(2i+1)a^{i+1}}\int_0^a\rho'\frac{d}{da'}(a'^{i+3}Y_i')da'$$

$$-\frac{a^i}{2i+1}\int_a^a\rho'\frac{d}{da'}\left(\frac{Y_i'}{a'^{i-2}}\right)da'=0, \quad \ldots\ldots (2)$$

except when $i=2$, in which case the second side is

$$\frac{m}{6}\frac{a^2\phi(a)}{a^3}\left(\frac{1}{3}-\mu^2\right) \quad \ldots\ldots\ldots\ldots\ldots (3).$$

By these equations Y_i is to be calculated, and then the form of the stratum of which the mean radius is a is known by the formula

$$r=a(1+Y_1+Y_2+\ldots+Y_i+\ldots).$$

PROP. *To prove that* $Y_i=0$, *excepting the case of* $i=2$.

76. Since Y_i and ρ are functions of a, they may be expanded into ascending series of the form

$$Y_i=Wa^s+\ldots, \quad \rho=D+D'a^n+\ldots,$$

where D is the density at the centre of the earth, and is as well as W and D' independent of a : s, n ... must not be negative, otherwise Y_i and ρ would be infinite at the centre.

Now when these and the corresponding series obtained by putting a' for a, are substituted in the equation of the strata in the last Article, and the first side arranged in powers of a, the various coefficients ought to vanish; excepting when $i=2$, because then the second side is not zero. We shall therefore substitute these series, and search for values of W and s which satisfy the condition.

$$\phi(a)=3\int_0^a\rho' a'^2da'=Da^3+\frac{3D'}{n+3}a^{n+3}+\ldots$$

After two easy integrations the equation of the strata becomes

$$WD\left(\frac{1}{3}a^{s+2}-\frac{a^{s-i+2}}{2i+1}a^{i}\right)+\ldots=0.$$

No value of s will cause these terms to vanish. The only apparent case is when $i=1$, for then by putting $s=i-2$ the part in the brackets vanishes: but in this particular case $s=-1$, and is negative and therefore inadmissible.

Hence the only way of satisfying the condition is by putting $W=0$; this shows that Y_i has no first term, that is, that it has no term at all and is therefore zero.

PROP. *To prove that the strata are all spheroidal, concentric, and have a common axis.*

77. By the last two Articles it appears that the equation to the surface $r=a(1+Y_2)$, and the equation for calculating Y_2 is

$$\frac{\phi(a)}{3a}Y_2-\frac{1}{5a^3}\int_0^a\rho'\frac{d}{da'}(a'^5Y_2')da'-\frac{a^2}{5}\int_a^a\rho'\frac{dY_2'}{da'}da'$$
$$=\frac{m}{6}\frac{a^2\phi(a)}{a^3}\left(\frac{1}{3}-\mu^2\right).$$

Suppose Y_2 (and similarly Y_2') is expanded in a series of powers of $\frac{1}{3}-\mu^2$, with indeterminate coefficients, to be ascertained by the condition that they shall satisfy the above equation. These coefficients will be functions of a only, as it is seen from the right-hand side of the equation that ω does not enter into the value of Y_2. It is clear that Y_2 consists of only one term, that involving the simple power of $\frac{1}{3}-\mu^2$. Let it be $\epsilon(\frac{1}{3}-\mu^2)$, ϵ being a small quantity of the order of m. Hence

$$r=a\{1+\epsilon(\tfrac{1}{3}-\mu^2)\},\ \mu=\sin(\text{latitude})=\sin l$$
$$=a(1-\tfrac{2}{3}\epsilon)(1+\epsilon\cos^2 l),\ \text{since } \epsilon \text{ is small.}$$

This is the equation to a spheroid from the centre, ϵ being the ellipticity. The axis-minor coincides with the axis of revolution of the whole mass. Hence the strata are concen-

tric spheroids, the minor-axes of which coincide with the axis of revolution of the whole mass.

78. The fluid theory therefore teaches us (1) that gravity is everywhere perpendicular to the surface (Art. 75); (2) that the exterior surface and the surfaces of the strata are all concentric spheroids, with the axis of each coincident with the axis of the earth (Art. 77).

These spheroids are of a definite form, depending upon the velocity of rotation and the law of density, as we shall see in Art. 80, and when we come to calculate their ellipticity. These spheroids so determined are called "spheroids of equilibrium," because they are the forms which the mass will assume when in equilibrium if it be fluid throughout. This term is used whether the mass has subsequently become solid or not, and refers solely to the form, not the condition. And in general when we say a body has a "surface of equilibrium," we mean that the surface though solid would retain its form if it became fluid, all other things remaining the same.

79. Since the strata are all concentric spheroids with a common axis in the axis of rotation, it follows that the centre of the earth's mass coincides with the centre of the volume, and that the axis of rotation is one of the principal axes of the mass. For this is true of each of the spheroidal strata separately, and is therefore true also of the aggregate or the whole mass.

PROP. *To obtain an equation for calculating the ellipticity of the strata.*

80. Substitute $\epsilon\,(\frac{1}{3}-\mu^2)$ for Y_2 and $\epsilon'\,(\frac{1}{3}-\mu^2)$ for Y_2' in equation (3) of Art. 75, and we have, after dividing by $\frac{1}{3}-\mu^2$,

$$\frac{\phi(a)}{3a}\epsilon - \frac{1}{5a^3}\int_0^a \rho' \frac{d}{da'}(a'^5\epsilon')\,da' - \frac{a^2}{5}\int_a^a \rho' \frac{d\epsilon'}{da'}da' = \frac{m}{6}\frac{a^2\phi(a)}{a^3}.$$

Divide both sides by a^2, and differentiate with respect to a; then multiply by a^6, and differentiate again, and divide by the coefficient of $\frac{d^2\epsilon}{da^2}$;

$$\therefore \frac{d^2\epsilon}{da^2} + \frac{6\rho a^2}{\phi(a)}\frac{d\epsilon}{da} - \left\{1 - \frac{\rho a^3}{\phi(a)}\right\}\frac{6\epsilon}{a^2} = 0.$$

This may be put into another form. Multiply by $\phi(a)$, then

$$\frac{d}{da}\left\{\phi(a)\frac{d\epsilon}{da}\right\} + \frac{d}{da}\{3\rho a^2\epsilon\} = \frac{6}{a^2}\phi(a)\,\epsilon + 3a^2\epsilon\frac{d\rho}{da};$$

$$\text{or } \frac{d^2}{da^2}\{\phi(a)\,\epsilon\} = \frac{6}{a^2}\phi(a)\,\epsilon + 3a^2\epsilon\frac{d\rho}{da}.$$

81. Cor. 1. By putting $a = \mathrm{a}$ in equation (3) of Art. 75, we have the following equation, which we shall find of use;

$$\int_0^{\mathrm{a}} \rho' \frac{d}{da'}(a'^5\epsilon')\,da' = \frac{5}{3}\mathrm{a}^2\phi(\mathrm{a})\left(\epsilon - \frac{m}{2}\right).$$

Prop. *To prove that the ellipticity of the strata decreases from the surface towards the centre.*

82. We assume that the density of the Earth increases from the surface to the centre. Let then $\rho = D - Ea^n + \ldots$, where E is positive: and $\epsilon = A + Ba^m + \ldots$. Then

$$\frac{\rho a^3}{\phi(a)} = 1 - \frac{n}{n+3}\frac{E}{D}a^n + \ldots = 1 - Ha^n + \ldots,\ H \text{ positive.}$$

Put these in the differential equation in ϵ of Art. 80; it gives

$$B(m^2+5m)\,a^{m-2} - 6AHa^{n-2} + \ldots = 0.$$

Neither m nor B can equal zero, because then the second term of ϵ only merges into the first. Nor can $m = -5$, a negative quantity. Hence the first term will not vanish of itself. But we may make the first and second vanish together by putting $n = m$ and $B(m^2+5m) = 6AH$. Hence B must be positive. And therefore *near the centre* ϵ increases towards the surface.

In thus increasing, suppose it attains a *maximum*, and then decreases. At this point $\frac{d\epsilon}{da} = 0$; and the equation of Art. 80, already used, gives

$$\frac{d^2\epsilon}{da^2} = \left\{1 - \frac{\rho a^3}{\phi(a)}\right\} \frac{6\epsilon}{a^2}, \text{ a } \textit{positive} \text{ quantity.}$$

This corresponds to a *minimum*. Hence ϵ does not attain a maximum, and therefore it continually increases from the centre to the surface. In the above we have assumed that $\phi(a)$ is greater than ρa^3. This appears

$$\because \phi(a) = 3\int_0^a \rho' a'^2 da = \rho a^3 - \int_0^a a'^3 \frac{d\rho'}{da'} da',$$

and $\frac{d\rho'}{da'}$ is negative by hypothesis.

83. The following Proposition, the converse of that of Art. 77, is of considerable importance, as it leads to the conclusion, that the form of the surface of the earth and the arrangement of the earth's mass are intimately related the one to the other. The opposite of this has sometimes been stated. For example, it has been said, that the law of gravity at the surface of the earth can be obtained theoretically without any reference to the arrangement of the mass. That this is erroneous will appear from what follows.

PROP. *To prove that if the form of the Earth's surface be a spheroid-of-equilibrium, the earth's mass must necessarily be arranged according to the fluid law, whether the mass is or has been fluid or not, in part or in whole.*

84. The meaning of a "spheroid of equilibrium" has been explained in Art. 78, and also of a "surface of equilibrium" in general. In consequence of the first property stated in that Article, a surface of equilibrium may be also thus defined. It is such that the resultant force at every point of the surface is at right angles to the surface at that point.

Now suppose some change were to be made in the arrangement of the earth's mass, without altering its external form. It is evident that, although the resultant attraction of the whole mass on the surface might possibly be unaltered by this change at particular points of the surface, it could not remain the same as before at every point of the surface.

Hence on this change being made in the internal arrangement, however slight it might be, the surface would cease to be one of equilibrium. In fact, if it were fluid it would at once assume another form, consequent on the internal change in the arrangement of the mass. Hence the form of the surface, if it be a surface-of-equilibrium, depends upon the arrangement of the mass. Suppose the arrangement of the mass throughout its solid and fluid parts follows that of the fluid law. That is, suppose that, not only the external surface is (what our hypothesis assumes it to be) a "spheroid of equilibrium," but that all the interior mass, whether solid or fluid, follows the fluid law of density. It is evident that in this case the surface would retain its form, even if the whole mass were to become fluid. Here is, then, one arrangement of the mass which we know accords with the form of the surface. And by what is said above it appears, that there can be but one such arrangement; as any departure from it will alter the attraction at the surface, and therefore deprive the surface of its character of being one of equilibrium. This, therefore, viz. the fluid law, is the only possible law of arrangement of the interior mass, if we know that the surface is a spheroid of equilibrium.

PROP. *To find the potential of the earth for an external point, on the hypothesis of the arrangement of the mass being according to the fluid law.*

85. By putting $a = \mathrm{a}$ in the formula for the potential of the earth given in Art. 75, it becomes, for an external point, bearing in mind Art. 76,

$$V = \frac{4\pi}{r}\int_0^{\mathrm{a}} \rho' \left\{ a'^2 + \frac{d}{da'}\left(\frac{a'^5}{5r^2}\, Y_2'\right)\right\} da',$$

$$= \frac{4\pi}{r}\int_0^{\mathrm{a}} \rho' \left\{ a'^2 + \frac{1}{5r^2}\,\frac{d\,(a'^5\epsilon')}{da'}\,(\tfrac{1}{3} - \mu^2)\right\} da', \text{ by Art. 77.}$$

Also by Art. 81,

$$\int_0^{\mathrm{a}} \rho' \,\frac{d\,(a'^5\epsilon')}{da'}\, da' = \frac{5}{3}\,\mathrm{a}^2\phi\,(\mathrm{a})\left(\epsilon - \frac{m}{2}\right) = \frac{5}{4\pi}\,\mathrm{a}^2 E\left(\epsilon - \frac{m}{2}\right);$$

$$\therefore\ V = \frac{E}{r} + \left(\epsilon - \frac{m}{2}\right)\frac{Ea^2}{r^3}\left(\frac{1}{3} - \mu^2\right),$$

for an external point, E being the mass.

86. Professor Stokes has obtained this formula in a somewhat different manner, which we introduce below on account of its elegance. He also deduces the results of Art. 79 regarding the centre and axis of the earth. But his investigation, in neither case, is more general than Laplace's theory here developed. For he assumes, not only that the mass is arranged in concentric strata all nearly spherical (see the course of the following investigation); but that the surface is one of equilibrium and also spheroidal, assumptions which, it will be seen from what goes before (see Art. 78), involve the whole fluid hypothesis, and need proof such as we have given above. The following is taken from his demonstration in the *Cambridge Philosophical Transactions* for 1849.

87. Let V be the potential of the mass. Then because the surface is a surface of equilibrium, (see Art. 75),

$$\text{const.} = V + \tfrac{1}{2}\omega^2 (1 - \mu^2) r^2.$$

By Art. 24 we have, for an external point,

$$r\frac{d^2.rV}{dr^2} + \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dV}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2V}{d\omega^2} = 0.$$

Let V be expanded in a series of Laplace's Functions,

$$V_0 + V_1 + \ldots + V_i + \ldots$$

Then since the above equation is linear with respect to V, and a series of Laplace's Functions cannot equal zero unless the Functions are separately zero (see Art. 35), we have, by substituting the above series for V and remembering the condition given by Laplace's Equation,

$$r\frac{d^2.rV_i}{dr^2} - i(i+1)\ V_i = 0.$$

Multiply by r^{-i-1} and integrate;

$$\therefore\ r^{-i}\frac{d(rV_i)}{dr} + ir^{-i-1}(rV_i) = \text{const.} = (2i+1)Z_i \text{ suppose.}$$

Multiply by r^{2i} and integrate;

$$\therefore\ r^i (r V_i) = Z_i r^{2i+1} + W_i,$$

$$V_i = \frac{W_i}{r^{i+1}} + Z_i r^i,$$

where W_i and Z_i are independent of r. The complete value of V becomes

$$V = \frac{W_0}{r} + \frac{W_1}{r^2} + \frac{W_2}{r^3} + \ldots + Z_0 + r Z_1 + r^2 Z_2 + \ldots$$

Now V evidently vanishes, from its very definition, when r is infinite. Hence $Z_0 = 0$, $Z_1 = 0$, $Z_2 = 0$ ...

$$\therefore\ V = \frac{W_0}{r} + \frac{W_1}{r^2} + \frac{W_2}{r^3} + \ldots$$

If the earth's strata* were exactly spherical instead of being nearly so, as shown in Art. 74, this expression would be reduced to its first term. Hence in our case W_1, W_2... must be all small quantities of the first and higher orders.

Substitute this in the equation of equilibrium and put

$$r = \mathrm{a}\{1 + \epsilon(\tfrac{1}{3} - \mu^2)\},\ \text{(Art. 77)};$$

$$\therefore\ \text{const.} = \frac{W_0}{\mathrm{a}}\{1 - \epsilon(\tfrac{1}{3} - \mu^2)\} + \frac{W_1}{\mathrm{a}^2} + \frac{W_2}{\mathrm{a}^3} + \ldots + \frac{1}{2} w^2 \mathrm{a}^2 (1 - \mu^2).$$

Equate the sums of Laplace's Functions of the same order to zero;

$$\therefore\ W_0 = \text{const.} - \tfrac{1}{3} w^2 \mathrm{a}^3,\ W_1 = 0,$$

$$W_2 = (W_0 \epsilon - \tfrac{1}{2} w^2 \mathrm{a}^3)\, \mathrm{a}^2 (\tfrac{1}{3} - \mu^2),\ W_3 = 0,\ W_4 = 0 \ldots$$

$$\therefore\ V = \frac{W_0}{r} + (W_0 \epsilon - \tfrac{1}{2} w^2 \mathrm{a}^3) \frac{\mathrm{a}^2}{r^3} (\tfrac{1}{3} - \mu^2).$$

* Professor Stokes's words are simply "If the surface were spherical"; but this is not sufficient. The surface of a body may be spherical and yet its mass so arranged that its potential on an external point will not be the mass divided by the distance from the centre.

W_0 is evidently equal to the mass, because as r becomes infinitely great the second term vanishes with reference to the first, and we know that in that case the value of the potential must be the mass divided by the distance. Let $W_0 = E$. Also put $m = w^2 . a^3 \div E$, as in Art. 75;

$$\therefore\ V = \frac{E}{r} + \left(\epsilon - \frac{m}{2}\right)\frac{Ea^2}{r^3}\left(\frac{1}{3} - \mu^2\right),$$

the same formula as before. It must always be borne in mind (what has been implied in the enunciation of the Prop.) that this formula is true only on the hypothesis of the fluid arrangement of the earth's mass being the real state of the earth.

§ 3. *Tests of the truth of the Fluid Theory of the Earth.*

88. There are four means of testing the truth of the Fluid Theory of the Earth, which we will now proceed to consider.

First Test. The Law of Gravity upon the surface to which the fluid theory leads; this law can be ascertained with great exactness by means of pendulum experiments.

Second Test. The amount which it assigns to the perturbation of the Moon's motion in Latitude caused by the distribution of the earth's mass in layers which are not spherical; the true amount has been found by astronomical observations.

Third Test. The Ellipticity of the Surface which the theory gives; this is determined by geodesy, as will be set forth in the last Chapter.

Fourth Test. The amount of Precession of the Equinoxes to which the theory leads; the true amount is known by observation.

The investigation on the fluid theory has proceeded thus far without making a single hypothesis—except, of course, the fundamental one which it is our object to test, viz. that the earth was once a fluid mass. The further investigation on which we now enter proceeds also without any hypothesis being made in applying the theory to the first two of these four tests, viz. the law of gravity and the moon's perturba-

tions. It is not till we come to the last two, viz. the ellipticity and the precession, that an hypothesis has to be introduced, viz. one regarding the Law of Density of the fluid mass. We therefore take these two tests last of the four.

All four of these tests bear important and independent testimony to the truth of the hypothesis that the earth was once a fluid mass.

First Test. THE LAW OF GRAVITY AT THE EARTH'S SURFACE.

89. Upon the hypothesis of the earth being a fluid mass it was shown by Clairaut, in his celebrated work, *Figure de la Terre*, published in 1743, that the increase of gravity in passing from the equator to the poles varies as the square of the sine of the latitude, and that a certain relation must necessarily subsist between the ellipticity and the amount of gravity, a relation which has been ever since known as *Clairaut's Theorem*. Laplace demonstrated the same, on the simpler hypothesis of the surface only being a surface of equilibrium, and the interior being solid or fluid, but consisting of strata nearly spherical. This we have shown in Art. 74 is a necessary condition of the earth.

Clairaut's Theorem is valuable as it enables us to determine the ellipticity by means of pendulum oscillations, the times of which measure the force of gravity at the several stations where experiments are made, and the result serves as a test of the correctness of the ellipticity deduced by the fluid theory.

PROP. *To find the law of gravity at the surface of a spheroid of equilibrium and of small ellipticity; and to prove* Clairaut's Theorem.

90. The potential of the earth for an external point is

$$V = \frac{E}{r} + \left(\epsilon - \frac{m}{2}\right)\frac{Ea^2}{r^3}\left(\frac{1}{3} - \mu^2\right), \text{ (Art. 85).}$$

Let g be gravity. Then since the angle between the radius vector r and the normal varies as the ellipticity and therefore its cosine must be taken = 1, the value of gravity is $-\frac{dV}{dr}$ — the part of the centrifugal force resolved along r

$$= \frac{E}{r^2} + 3\left(\epsilon - \frac{m}{2}\right)\frac{Ea^2}{r^4}\left(\frac{1}{3} - \mu^2\right) - w^2 r\,(1-\mu^2)$$

$$= \frac{E}{r^2} + 3\left(\epsilon - \frac{m}{2}\right)\frac{Ea^2}{r^4}\left(\frac{1}{3} - \mu^2\right) - m\frac{E}{a^3}r\left(\frac{2}{3} + \frac{1}{3} - \mu^2\right).$$

Substitute for r and omit small quantities of the second order,

$$\therefore\ g = \frac{E}{a^2}\left(1 - \frac{2}{3}m\right) - \frac{E}{a^2}\left(\frac{5}{2}m - \epsilon\right)\left(\frac{1}{3} - \mu^2\right)$$

$$= \frac{E}{a^2}\left\{1 + \frac{1}{3}\epsilon - \frac{3}{2}m + \left(\frac{5}{2}m - \epsilon\right)\sin^2.\text{latitude}\right\}$$

$$= G\left\{1 + \left(\frac{5}{2}m - \epsilon\right)\sin^2 l\right\},$$

where G is gravity at the equator.

Hence the increase of gravity in passing from the equator to the poles varies as the square of the sine of the latitude.

This expression immediately leads to the following property, called *Clairaut's Theorem* after its first discoverer.

$$\frac{\text{Polar gravity} - \text{equatorial gravity}}{\text{equatorial gravity}} + \text{ellipticity}$$

$$= \left(\frac{5}{2}m - \epsilon\right) + \epsilon = \frac{5}{2}m$$

$$= \frac{5}{2} \times \text{ratio of centrifugal force at equator to gravity.}$$

91. When the formula of gravity deduced in the last Article is applied to the several stations where pendulum experiments have been made, discrepancies are brought to light which evidently arise, not from any mistake in the theory, but from the irregularities of the surface of the earth. Mr Airy, in his article on the Figure of the Earth, written in 1830, (*Ency. Met.*) discussed all the data which had been obtained in various places, and came to the following results: (1) that

gravity appears to be greater on islands than on continents; (2) that gravity is greater in high north latitudes, less in middle latitudes than the formula gives it, but is pretty nearly the same about the equator; (3) that gravity does not appear to vary with the longitude alone, nor the hemisphere.

Professor Stokes, in a paper in the *Cambridge Philosophical Transactions* for 1849, has fully discussed the various causes of disturbance, and has satisfactorily explained the anomalies (1) and (2) deduced by Mr Airy, twenty years earlier, from the experiments. He has also shown, that when allowance is made for the irregularities of the earth's surface, the ellipticity comes out somewhat smaller than it otherwise would.

The chief sources of error are the following. The elevation of the station above the sea-level, and the excess or defect of matter in table-lands or the sea.

The value of gravity obtained by pendulum experiments must be reduced to the standard of the sea-level, and corrected for that level in the way explained in Art. 53. But the sea-level, owing to local attraction, is higher in continents than at sea, as will appear in the next Chapter. Hence gravity obtained from continental experiments will be too small, because it is corrected for a surface too distant from the centre of the earth. This well explains why gravity appears to be less on continents than on islands. The same explanation meets the second anomaly pointed out by Mr Airy. In the middle latitudes the places where experiments were made are all continental. If this is corrected for, no doubt the deduced ellipticity will come out somewhat smaller, and therefore gravity in high latitudes, as deduced from the formula, no longer be in excess.

Mr Stokes remarks, that if the 49 stations where pendulum experiments have been made are divided into two groups, an equatorial group containing the stations lying between latitudes 35^0 N. and 35^0 S., and a polar group containing the rest, it will be found that most if not all of the oceanic stations are contained in the former group, while the stations belonging to the latter are of a more continental character. Hence the observations will make gravity appear too great about the equator and too small about the poles, that is, they will on the whole make

gravity vary too little from the equator to the poles; and since the variation depends on $\frac{5}{2}m - \epsilon$, the observations will be best satisfied by a value of ϵ which is too great. This is, in fact, precisely the result of the discussion; the value of ϵ which Mr Airy has obtained from pendulum experiments (0·003535) being, as stated, greater than that which he finds from the discussion of geodetic measures (0·003352).

92. A large collection of the results of pendulum experiments is to be found in Major-General Sabine's work entitled, *Account of Experiments to determine the Figure of the Earth by means of the Pendulum Vibrating Seconds in different Latitudes*, 1825.

The following abstract is taken from his translation of the *Cosmos*, Vol. IV. Part I. The column of "computed" vibrations assumes that the change of gravity varies as the change in the square of the sine of the latitude.

VIBRATIONS.

Stations.	Latitude.	Vibrations. Computed.	Vibrations. Observed.	Differences.
		s.	*s.*	*s.*
Equator	0° 0′ 0″	86263,60		
St Thomas	0 24 41 N.	86263,60	86269,32	+5,72
Maranham	2 31 34 S.	86264,30	86259,77	−4,53
Ascension	7 55 30 S.	86267,86	86273,04	+5,18
Sierra Leone	8 29 28 N.	86268,48	86268,33	−0,15
Trinidad	10 38 55 N.	86271,24	86267,27	−3,97
Bahia	12 59 21 S.	86274,90	86273,16	−1,74
Jamaica	17 56 7 N.	86284,80	86285,12	+0,32
New York	40 42 43 N.	86358,66	86357,73	−0,93
Paris	48 50 14 N.	86390,20	86388,48	−1,72
Shanklin	50 37 24 N.	86397,06	86396,54	−0,52
Greenwich	51 28 40 N.	86400,34	86400,59	+0,25
London	51 31 8 N.	86400,48	86400,00	−0,48
Arbury	52 12 55 N.	86403,12	86403,31	+0,19
Clifton	53 27 43 N.	86407,80	86407,23	−0,57
Altona	35 32 45 N.	86408,10	86408,94	+0,84
Leith	55 58 41 N.	86417,02	86417,89	+0,87
Portsoy	57 40 59 N.	86423,10	86424,60	+1,50
Unst	60 45 28 N.	86433,64	86435,56	+1,92
Drontheim	63 25 54 N.	86442,24	86438,77	−3,47
Hammerfest	70 40 5 N.	86462,42	86461,05	−1,37
Greenland	74 32 19 N.	86471,00	86470,50	−0,50
Spitzbergen	79 49 54 N.	86479,90	86483,01	+3,11

The ratio of increase of gravity from the equator to the pole deduced from these observations is 0·0051828, (see *Cosmos*, p. 468.)

PROP. *To find the ellipticity by Clairaut's Theorem and Pendulum experiments.*

93. By Clairaut's Theorem and the last Article

$$\epsilon = \frac{5}{2} m - 0{\cdot}0051828 = 0{\cdot}0086355 - 0{\cdot}0051828$$

$$= 0{\cdot}0034527 = \frac{1}{289}.$$

The investigations of Professor Stokes referred to in Art. 91, show that this should be a little smaller.

The actual measurement of the earth, as we shall show in the next Chapter, makes the ellipticity $\frac{1}{294}$. Hence pendulum experiments bear a strong testimony in favour of the fluid theory.

But besides the value of the ellipticity deduced from pendulum experiments, the law of variation of gravity on the surface as deduced above from the fluid theory, viz. that it changes as the square of the latitude, also agrees remarkably with those experiments, and bears strong testimony to the truth of that theory.

94. In addition to the general test of the truth of the fluid arrangement thus afforded by pendulum experiments we will add an investigation which still further tests the truth of the theory, by finding what effect certain hypothetical redistributions of the mass, differing from the fluid arrangement, would have upon the motion of the pendulum. If the effect would be sensible we have a further argument in favour of the fluid arrangement being the actual arrangement of the earth's mass.

PROP. *To find the effect on the pendulum of certain hypothetical changes in the distribution of the materials of the earth's mass.*

95. We will suppose the earth's mass divided into four shells and a nucleus, the radius of the nucleus and the thickness of each shell being equal to one-fifth of the earth's radius, or about 800 miles. We shall make three separate hypotheses:—

(1) That the masses of the second and third shells are both altered, each in a different proportion, so as to preserve the whole mass the same and not to alter the form of the strata.

(2) That the form of the strata in one of the shells is altered without affecting the mass.

(3) That the earth consists of a homogeneous mass of the same density as the surface, with the remainder of the mass distributed according to any law in spherical shells.

96. *First re-arrangement.* Let $EE_1E_2E_3E_4$ be the masses of the earth and of the portions of it lying within the inner surfaces of the four successive shells: $VV_1V_2V_3V_4$ the corresponding potentials for a point at distance r from the centre and in latitude of which the sine is μ. Then, by Art. 85,

$$V = \frac{E}{r} + \left(\epsilon - \frac{m}{2}\right)\frac{Ea^2}{r^3}\left(\frac{1}{3} - \mu^2\right),$$

and $V_1V_2\ldots$ have corresponding values, $\epsilon_1\epsilon_2\ldots$ $m_1m_2\ldots$ being similar quantities to ϵ and m. Then $V_1 - V_2$ and $V_2 - V_3$ are the potentials of the second and third shells. Also $E_1 - E_2$, $E_2 - E_3$ are the masses of those shells. Suppose the first of these masses is altered in the ratio $\alpha : 1$, and the second in the ratio $\beta : 1$; then as the total mass is unaltered, by hypothesis,

$$\alpha(E_1 - E_2) + \beta(E_2 - E_3) = E_1 - E_3,$$

$$\text{or } 1 - \beta = (\alpha - 1)\frac{E_1 - E_2}{E_2 - E_3}.$$

In consequence of this change the potentials of the shells become $\alpha(V_1 - V_2)$ and $\beta(V_2 - V_3)$. Hence if U be the potential of the whole earth thus altered in the arrangement of its materials

$$U = V + (\alpha - 1)(V_1 - V_2) + (\beta - 1)(V_2 - V_3)$$

$$= \frac{E}{r} + \frac{E a^2}{r^3}\left[\epsilon - \frac{m}{2} + (\alpha - 1)\left\{\frac{E_1}{E}\frac{a_1^2}{a^2}\left(\epsilon_1 - \frac{m_1}{2}\right) - \frac{E_2}{E}\frac{a_2^2}{a^2}\left(\epsilon_2 - \frac{m_2}{2}\right)\right\}\right.$$

$$\left. + (\beta - 1)\left\{\frac{E_2}{E}\frac{a_2^2}{a^2}\left(\epsilon_2 - \frac{m_2}{2}\right) - \frac{E_3}{E}\frac{a_3^2}{a^2}\left(\epsilon_3 - \frac{m_3}{2}\right)\right\}\right]\left(\frac{1}{3} - \mu^2\right).$$

As m is the ratio of the centrifugal force to gravity at the equator of the spheroid,

$$\therefore \frac{m_1}{m} = \frac{E}{E_1}\frac{a_1^3}{a^3}, \quad \frac{m_2}{m} = \frac{E}{E_2}\frac{a_2^3}{a^3}, \quad \frac{m_3}{m} = \frac{E}{E_3}\frac{a_3^3}{a^3};$$

$$\therefore U = \frac{E}{r} + \frac{E a^2}{r^3}\left\{\epsilon - \frac{m}{2} + L\right\}\left(\frac{1}{3} - \mu^2\right),$$

where $L =$

$$(\alpha - 1)\left\{\frac{E_1}{E}\frac{a_1^2}{a^2}\epsilon_1 - \frac{E_2}{E}\frac{a_2^2}{a^2}\epsilon_2\frac{E_1 - E_3}{E_2 - E_3} + \frac{E_3}{E}\frac{a_3^2}{a^2}\epsilon_3\frac{E_1 - E_2}{E_2 - E_3}\right.$$

$$\left. - \frac{m}{2}\left(\frac{a_1^5}{a^5} - \frac{a_2^5}{a^5}\frac{E_1 - E_3}{E_2 - E_3} + \frac{a_3^5}{a^5}\frac{E_1 - E_2}{E_2 - E_3}\right)\right\}.$$

By substituting the values found in future Articles (Art. 107, 8), according to the fluid-hypothesis, we have

$$L = (\alpha - 1)\,\epsilon\left(\frac{0{\cdot}6628}{1{\cdot}09}\frac{16}{25} - \frac{0{\cdot}3371}{1{\cdot}16}\frac{9}{25}\frac{5494}{2237} + \frac{0{\cdot}1134}{1{\cdot}20}\frac{4}{25}\frac{3257}{2237}\right)$$

$$- (\alpha - 1)\frac{m}{2}\left(\frac{1024}{3125} - \frac{243}{3125}\frac{5494}{2237} + \frac{32}{3125}\frac{3257}{2237}\right)$$

$$= (\alpha - 1)\left(\frac{0{\cdot}3892 - 0{\cdot}2569 + 0{\cdot}0220}{294}\right.$$

$$\left. - \frac{0{\cdot}3277 - 0{\cdot}1910 + 0{\cdot}0149}{578}\right)$$

$$= (\alpha - 1)\,(0{\cdot}0005316 - 0{\cdot}0002623) = (\alpha - 1) \times 0{\cdot}0002693,$$

the value of ϵ here used being taken from the British Ordnance Survey.

Now gravity $= -\frac{dV}{dr}$ – centrifugal force, at the surface.

Hence the ratio of gravity as altered by this change to gravity as it is

$$= \left(-\frac{dU}{dr} - \text{centrifugal force}\right) \div \left(-\frac{dV}{dr} - \text{centrifugal force}\right)$$

$$= 1 + 3L\left(\frac{1}{3} - \mu^2\right),$$

and the increase in passing from the equator to the pole

$$= 3L = (\alpha - 1) \times 0{\cdot}0008079$$

$$= (\alpha - 1)\frac{0{\cdot}0008079}{0{\cdot}0051828} \times \text{actual increase (see Art. 92)},$$

$$= \frac{\alpha - 1}{6{\cdot}4} \times \text{actual increase.}$$

The table in Art. 92 shows that between the equator and Spitzbergen in about 80° north latitude (the highest place north where pendulum experiments have been made) 214 vibrations are lost in 24 hours by a seconds' pendulum. Hence the number which would be lost from the re-arrangement of the mass now under consideration would equal

$$\frac{\alpha - 1}{6{\cdot}4} \times 214 = 33\,(\alpha - 1).$$

Suppose that a difference of 5 beats of the pendulum at the equator and at Spitzbergen is easily detected, then if

$$33\,(\alpha - 1) = 5, \text{ or } \alpha = 1 + \frac{1}{7}, \text{ and } \beta = 1 - \frac{1}{7} \times \frac{3257}{2237} = 1 - \frac{1}{5}$$

nearly, that is, if the density of the second shell be increased about 1-7th and that of the third be diminished by about 1-5th, the deranging effect on the pendulum would be capable of detection on the earth's surface. Under these circumstances it may be said, that the near approach to conformity between the observed number of vibrations of the pendulum in

various places and the same computed on the fluid-arrangement of the mass affords some argument in favour of that arrangement representing the actual condition of the earth, whether it be now in part fluid or not.

97. *Second re-arrangement.* Suppose that all the strata in one of the shells lose their ellipticity and become spherical, the parts about the poles of the upper surface of the shell swelling up and penetrating the mass of the shell above it, and the parts about the equator of the lower surface of the shell penetrating the shell within; so that the original spheroidal shell may become a spherical shell of the same mass as before, and its mass still co-existing with the other shells and nucleus. This change amounts simply to this: the density of the mass is doubled through a thin space of the form of two hemispherical meniscuses, the rims of which are of no thickness and touch each other at the equator of the upper surface of the shell in question, the thickness of the meniscuses being greatest at the poles and equalling the compression of that surface; and the density is also doubled through a space at the lower surface of the shell generated by the revolution of a crescent round the earth's axis of which the width at the middle equals the distance of the equator of that lower surface from the inscribed sphere. The matter causing the doubling of the density through these thin spaces is drawn from the original shell itself.

We will take the second shell for our example, and will apply the result to the other shells.

The potential of this shell is $V_1 - V_2$; this must now be replaced by $(E_1 - E_2) \div r$, the potential of a shell of spherical strata. Hence

$$U = V - (V_1 - V_2) + (E_1 - E_2) \div r$$

$$= \frac{E}{r} + \frac{E a^2}{r^3} \left\{ \epsilon - \frac{m}{2} - \left(\epsilon_1 - \frac{m_1}{2} \right) \frac{E_1}{E} \frac{a_1^2}{a^2} + \left(\epsilon_2 - \frac{m_2}{2} \right) \frac{E_2}{E} \frac{a_2^2}{a^2} \right\} \left(\frac{1}{3} - \mu^2 \right)$$

$$= \frac{E}{r} + \frac{E a^2}{r^3}\left\{\epsilon - \frac{m}{2} - \frac{E_1}{E}\frac{a_1^2}{a^2}\epsilon_1 + \frac{E_2}{E}\frac{a_2^2}{a^2}\epsilon_2 + \frac{m}{2}\frac{a_1^5 - a_2^5}{a^5}\right\}\left(\frac{1}{3} - \mu^2\right)$$

$$= \frac{E}{r} + \frac{E a^2}{r^3}\left\{\epsilon - \frac{m}{2} - N\right\}\left(\frac{1}{3} - \mu^2\right)$$

$$\text{where } N = \frac{E_1}{E}\frac{a_1^2}{a^2}\epsilon_1 - \frac{E_2}{E}\frac{a_2^2}{a^2}\epsilon_2 - \frac{m}{2}\frac{a_1^5 - a_2^5}{a^5},$$

and, as before, $3N$ is the whole decrease which would thus be produced in gravity from the equator to the poles by this change of distribution of the mass. We shall calculate N for the second, third, and fourth shells.

$$N = \frac{1}{294}\left(\frac{0{\cdot}6628}{1{\cdot}09}\frac{16}{25} - \frac{0{\cdot}3371}{1{\cdot}16}\frac{9}{25}\right) - \frac{1}{578}\frac{1024-243}{3125},$$

$$\frac{1}{294}\left(\frac{0{\cdot}3371}{1{\cdot}16}\frac{9}{25} - \frac{0{\cdot}1134}{1{\cdot}20}\frac{4}{25}\right) - \frac{1}{578}\frac{243-32}{3125},$$

$$\frac{1}{294}\left(\frac{0{\cdot}1134}{1{\cdot}20}\frac{4}{25} - \frac{0{\cdot}0153}{1{\cdot}23}\frac{1}{25}\right) - \frac{1}{578}\frac{32-1}{3125},$$

$$= 0{\cdot}0005354,\ 0{\cdot}0001876,\ 0{\cdot}0000322.$$

The ratios these, multiplied by 3, bear to the actual increase of gravity are

$$0{\cdot}310,\ 0{\cdot}167,\ 0{\cdot}019.$$

And therefore the number of beats gained at Spitzbergen upon the pendulum at the equator would be 214 multiplied by these fractions, or 66, 23, and 4. The first and second of these might be detected, though not the third. This calculation shows that a comparatively small change in the form of the strata would have a very perceptible influence upon the pendulum. The effect is greater than is produced in the former case, by a mere change in densities without altering the form of the strata: and this tells very strongly in favour of the fluid-arrangement, and indeed of the fluid-theory itself. We might

perhaps conceive the *external* surface of an irregular mass revolving round a fixed axis assuming, after an enormous period, a generally spheroidal form, because the perpetual weathering of the surface would set free parts of the solid materials, which with the fluids would arrange themselves according to fluid principles. But the *interior* parts could not thus arrange themselves, as these calculations seem to show they have done, unless they had at one time been fluid or semi-fluid, so as to partake of that bulging form about the equator of each stratum which the motion of rotation tends to produce.

98. *Third re-arrangement.* The following is perhaps a still better hypothetical arrangement of the earth's mass with a view to testing the fluid theory of its origin. Suppose that the earth is a solid mass which, as described above, has acquired its external spheroidal form by the action of time; and imagine its mass to be made up of a homogeneous spheroid of the earth's present form, but of the density only of the surface, with the remainder of the mass distributed anyhow in spherical shells around the centre.

The density of the surface is half the mean density of the earth; hence the mass of the homogeneous spheroid will be half the whole mass;

$$\therefore\ U = \frac{\frac{1}{2}E}{r} + \frac{\frac{1}{2}Ea^2}{r^3}\left(\epsilon - \frac{m}{2}\right)\left(\frac{1}{3} - \mu^2\right) + \frac{\frac{1}{2}E}{r}$$

$$= \frac{E}{r} + \frac{1}{2}\frac{Ea^2}{r^3}\left(\epsilon - \frac{m}{2}\right)\left(\frac{1}{3} - \mu^2\right),$$

and the consequent increase of gravity between the equator and the poles will

$$= \frac{3}{2}\left(\epsilon - \frac{1}{2}m\right),$$

or, when compared with the actual increase of gravity,

$$= \frac{3}{2}\left(\frac{1}{294} - \frac{1}{578}\right) \div 0{\cdot}0051828 = 0{\cdot}48.$$

This is nearly half the actual increase. Hence if this were the actual distribution, the gain of the pendulum over its rate

at the equator would everywhere be only about half what experiment makes it to be.

Experiment shows no sudden changes, nor any marked deviation from a regular increase, varying as the change in the square of the latitude, in the rates of the pendulum in passing from the equator towards the poles. Hence the excess of matter above the homogeneous spheroid cannot be distributed irregularly. We have supposed it to be distributed in spherical shells, and the change on the pendulum would be, as we have shown, very great, and would be very perceptible indeed. Any departure from the spherical form, not towards the oblate spheroids required by the fluid-theory, but in the opposite direction, would produce a result still more discordant with experiment; whereas every approach in the distribution to those spheroids will bring the calculation into nearer accordance with fact. No stronger testimony can well be borne to the truth of the fluid-arrangement and fluid-theory.

99. Cor. We may find the effect of a large departure from regularity in the mass in the following manner. Suppose there is a preponderance of matter the effect of which may be represented by a spherical mass m, the distance of the centre of which from the centre is c, $= x.\mathrm{a}$ suppose. Then the difference of gravity in consequence of this at the two points nearest and furthest off from this mass

$$= \frac{m}{(\mathrm{a}-c)^2} - \frac{m}{(\mathrm{a}+c)^2} = \frac{4\mathrm{a}cm}{(\mathrm{a}^2-c^2)^2} = \frac{4xm}{\mathrm{a}^2(1-x^2)^2},$$

the ratio of this to gravity

$$= \frac{m}{E}\frac{4x}{(1-x^2)^2}.$$

If b be the number of beats lost or gained by a seconds' pendulum between the two places in 24 hours

$$\frac{2b}{24\times 60\times 60} = \frac{m}{E}\frac{4x}{(1-x^2)^2};$$

$$\therefore \frac{m}{E} \text{ or } \frac{e^3}{\mathrm{a}^3} = x\left(\frac{1}{x}-x\right)^2 \times 0{\cdot}00000579b,$$

where e is the radius of a sphere of which the density is the mean density of the earth and the mass $= m$.

Ex. 1. Suppose $b = 10$, $x = \frac{3}{4}$, then $e = 96$ miles. That is, a mass of only radius 96 miles and only fourteen millionths and a half of the earth's mass, and as far down as 1000 miles from the surface, will have a perceptible influence upon the pendulum.

Ex. 2. If the depth below the surface be 500 miles the radius of the disturbing mass will be only 62 miles, and the mass three millionths and three-quarters of the earth's mass.

Ex. 3. If $x = \frac{1}{4}$, $e = 236$ miles and m is 1-5000th part of the earth's mass, and 3000 miles below the surface; and yet in each of these cases the effect is the same as before.

Accurate pendulum experiments all over the world must bring to light such masses, if they exist. None have as yet been detected.

Second Test. PERTURBATION OF THE MOON'S MOTION IN LATITUDE.

100. Laplace first pointed out that the ellipticity of the Earth would have an effect upon the Moon's motion. This we shall use as our second general test of the truth of the fluid-theory.

PROP. *To find the effect of the Earth's mass, arranged according to the fluid law, upon the Moon's motion in latitude.*

101. By the Planetary Theory (see *Mechanical Philosophy*, second edition, p. 329; or Cheyne's *Planetary Theory*, p. 35),

$$\frac{di}{dt} = \frac{na}{(E+M)\,i}\frac{dR}{d\Omega}, \quad \frac{d\Omega}{dt} = -\frac{na}{(E+M)\,i}\frac{\dot{R}}{di},$$

where n is the mean motion of the moon about the earth, a the mean distance, E and M the masses, i the inclination of the moon's orbit to the ecliptic (the square of which is neg-

lected), Ω the longitude of its node, R the disturbing function such that its differential coefficient with respect to any line drawn from the moon is the disturbing force acting on the moon in that direction, reckoned positive if acting on the side of the origin of the co-ordinates. If V be the potential of the earth with reference to the moon condensed into its centre, then $MV \div E$ will be the potential of the moon with reference to the earth; and in calculating the motion of the moon about the earth, we must imagine the earth reduced to rest by the moon's attraction being applied in an opposite direction to both the earth and moon. Hence the disturbing function R, which refers to the difference of attraction of the earth's mass as condensed into its centre and as arranged according to the fluid-law,

$$= \frac{E+M}{E} V - \frac{E+M}{r} = \left(\epsilon - \frac{m}{2}\right) \frac{(E+M)\, a^2}{r^3} \left(\frac{1}{3} - \mu^2\right),$$

r being the distance of the moon from the earth's centre. Let λ and θ be the latitude and longitude of the moon, ϵ' the epoch, ϖ the longitude of the perigee, I the obliquity of the ecliptic. Then, as λ and i are both small,

$$\tan \lambda = \tan i \sin (nt + \epsilon' - \Omega), \text{ or } \lambda = i \sin (nt + \epsilon' - \Omega).$$

$$\text{Also } \theta = nt + \epsilon' + 2e \sin (nt + \epsilon' - \varpi);$$

$$\therefore \mu = \cos (\text{moon's north polar distance})$$

$$= \sin I \cos \lambda \sin \theta + \cos I \sin \lambda$$

$$= \sin I \sin (nt + \epsilon') + i \cos I \sin (nt + \epsilon' - \Omega)$$

$$- e \sin I \sin \varpi + e \sin I \sin (2nt + 2\epsilon' - \varpi).$$

Substituting this in R, and preserving only the terms which are periodical and also independent of $nt + \epsilon'$, since these last go through their changes so rapidly as to neutralize their effects very quickly, we have

$$R = \frac{(E+M)\, a^2}{2r^3} \left(\epsilon - \frac{m}{2}\right) i \sin 2I \cos \Omega = (E+M)\, A \,.\, i \cos \Omega;$$

$$\therefore \frac{di}{dt} = -naA \sin \Omega, \quad \frac{d\Omega}{dt} = -\frac{na}{i} A \cos \Omega.$$

Since the node on the whole retrogrades pretty steadily, we may put $\Omega = -ht$ on the second side of these equations, h being the mean regression. Hence δ being the symbol of variation in i and Ω,

$$\delta i = -\frac{na}{h} A \cos \Omega, \quad \delta\Omega = +\frac{na}{hi} A \sin \Omega;$$

$$\therefore \delta\lambda = \sin (nt + \epsilon' - \Omega)\, \delta i - i \cos (nt + \epsilon' - \Omega)\, \delta\Omega$$

$$= -\frac{na}{h} A \sin (nt + \epsilon')$$

$$= -\frac{na^2}{2ha^2} \left(\epsilon - \frac{m}{2}\right) \sin 2I \sin (nt + \epsilon'),$$

putting $r = a$. Burg makes this term

$$= -8'' \sin (nt + \epsilon')$$

by observation. Also $h = 0{\cdot}0040217n$. Hence after all reductions,

$$\epsilon - \frac{m}{2} = 0{\cdot}0015474, \text{ and } \frac{m}{2} = \frac{1}{578} = 0{\cdot}0017476;$$

$$\therefore \epsilon = 0{\cdot}0032950 = \frac{1}{303} \text{ nearly.}$$

This result differs but slightly from the measure obtained by geodesy; it is a little too small. But considering the minuteness of the quantity to be determined, the result is remarkable, and bears its testimony to the truth of the fluid-arrangement of the earth's mass.

Third Test. **THE ELLIPTICITY OF THE SURFACE.**

102. For the application of this and the next test the formulæ cannot be obtained without assuming a law of density in the distribution of the earth's mass.

PROP. *To find a law of density of the earth's mass.*

103. In order to make the equation in Art. 80 for determining the ellipticity integrable, it is necessary to assume a law of density of the mass of the earth. Experiment has not yet determined what the law of compression in such a mass would be.

In order to make the equation

$$\frac{d^2}{da^2}\{\phi(a)\,\epsilon\} = \frac{6}{a^2}\phi(a)\,\epsilon + 3a^2\epsilon\frac{d\rho}{da}$$

integrable, we must assume the last term to be some multiple of $\phi(a)\,\epsilon$, the other factor being a function of a. Suppose

$$3a^2\frac{d\rho}{da} = -q^2\phi(a) = -3q^2\int_0^a \rho' a'^2 da',$$

the negative sign being taken because the density decreases from the centre to the surface; q^2 is some function of a. The equation for the ellipticity becomes

$$\frac{d^2\{\phi(a)\,\epsilon\}}{da^2} + \left(q^2 - \frac{6}{a^2}\right)\phi(a)\,\epsilon = 0 \ldots\ldots\ldots\ldots(1).$$

The only case in which this equation has been integrated is when q^2 is constant. We must therefore assume it to be so*. The equation for the density then gives by differentiation and re-adjustment

$$\frac{d^2.\rho a}{da^2} + q^2.\rho a = 0;$$

$$\therefore\ \rho a = Q\sin(qa + A).$$

But ρ would be infinite at the centre unless $A = 0$. Hence

$$\rho = Q\frac{\sin qa}{a}.$$

* In order to explain how to differentiate a definite integral with respect to a quantity involved in the limits, let $\int f(x) = F(x) + \text{const.}$;

$$\therefore \int_b^a f(x)\,dx = F(a) - F(b);$$

$$\therefore \frac{d}{da}\int_b^a f(x)\,dx = \frac{dF(a)}{da} = f(a);\quad \frac{d}{db}\int_b^a f(x)\,dx = -\frac{dF(b)}{db} = -f(b).$$

In the case of the earth the constants Q and q must be found from the conditions, that the density at the surface is the density of granite; and that the mean density of the whole is twice that density. This last leads to the formula,

$$2 = \text{mean density} \div \text{superficial density}$$

$$= \text{mass} \div (\text{volume} \times \text{superficial density})$$

$$= 4\pi \int_0^a \rho a^2 \, da \div \left(\frac{4\pi}{3} a^3 . \, Q \frac{\sin qa}{a}\right)$$

$$= 3 \int_0^a a \sin qa \, da \div a^2 \sin qa = \frac{3}{q^2 a^2}\left(1 - \frac{qa}{\tan qa}\right).$$

This equation is satisfied by

$$qa = 2{\cdot}4605, \; = 140^\circ \, 58' \, 35''.$$

If D be the density of the surface or 2·75, then

$$Q = \frac{Da}{\sin qa} = \frac{2{\cdot}75a}{0{\cdot}63} = 4{\cdot}365a,$$

and also $Qq = 10{\cdot}74$.

104. If the law of density deduced in Art. 103 be used in equation (1) of Art. 75, then, neglecting the small term,

$$\int_0^a \frac{1}{\rho'} \frac{dp}{da'} da' = \frac{4\pi}{3} \frac{\phi(a)}{a} + 4\pi \int_a^a \rho' a' \, da'$$

$$= \frac{4\pi}{a} Q \int_0^a a' \sin qa' \, da' + 4\pi Q \int_a^a \sin qa' \, da'$$

$$= \frac{4\pi}{q} Q \left(\frac{\sin qa}{qa} - \cos qa - \cos qa + \cos qa\right)$$

$$= \frac{4\pi}{q^2} \rho - \frac{4\pi}{q} \cos qa.$$

Hence by differentiation with respect to a,

$$\frac{dp}{da} = \frac{4\pi}{q^2} \rho \, \frac{d\rho}{da} = \frac{2\pi}{q^2} \frac{d.\rho^2}{da},$$

or the increase of pressure varies as the increase in the square of the density. It was by assuming this law between pressure and density, that Laplace deduced the law of density which we have arrived at in another way, viz. by finding what condition is necessary to make the equation of ellipticities integrable (*Mémoires de l'Institut*, Tom. III. p. 496).

PROP. *To find an expression for the ellipticity of the strata, with the law of density deduced in the last Proposition but one, and to reduce it to numbers for the surface.*

105. In order to integrate the equation of ellipticities in Art. 80, put

$$\phi(a)\epsilon = \frac{1}{a^2}\int_0^a a' \int_0^{a'} a'x'da'^2;$$

$$\therefore \frac{d.\phi(a)\epsilon}{da} = -\frac{2}{a^3}\int_0^a a' \int_0^{a'} a'x'da'^2 + \frac{1}{a}\int_0^a a'x'da';$$

$$\therefore \frac{d^2.\phi(a)\epsilon}{da^2} = \frac{6}{a^4}\int_0^a a' \int_0^{a'} a'x'da'^2 - \frac{2}{a^3}\int_0^a a'x'da' - \frac{1}{a^2}\int_0^a a'x'da' + x;$$

$$\therefore x - \frac{3}{a^3}\int_0^a a'x'da' + \frac{q^2}{a^3}\int_0^a a' \int_0^{a'} a'x'da'^2 = 0.$$

Multiply by a^3 and differentiate;

$$\therefore a^3\frac{dx}{da} + 2ax - 3ax + q^2a\int_0^a a'x'da' = 0.$$

Divide by a and differentiate, and then divide by a;

$$\frac{d^2x}{da^2} + q^2x = 0.$$

The solution of this is

$$x + Cq^2\sin(qa + B) = 0,$$

C and B being independent of a;

$$\therefore \int_0^a a'x'da' = Cqa\cos(qa + B) - C\sin(qa + B);$$

$$\therefore\ \phi(a)\,\epsilon = \frac{1}{a^2}\left\{Ca^2\sin(qa+B) + \frac{2C}{q}a\cos(qa+B)\right.$$

$$\left. - \frac{2C}{q^2}\sin(qa+B) + \frac{C}{q}a\cos(qa+B) - \frac{C}{q^2}\sin(qa+B)\right\}$$

$$= C\left\{\left(1-\frac{3}{q^2a^2}\right)\sin(qa+B) + \frac{3}{qa}\cos(qa+B)\right\}.$$

In our case $B=0$, otherwise the ellipticity at the centre would be infinite, as is easily seen by expanding ϵ in powers of a.

Hence, if we substitute for $\phi(a)$, the ellipticity

$$= \frac{Cq^2}{3Q}\,\frac{\left(1-\frac{3}{q^2a^2}\right)\tan qa + \frac{3}{qa}}{\tan qa - qa}.$$

And the ratio of this to the ellipticity of the surface

$$= \frac{\tan qa - qa}{\tan qa - qa}\,\frac{\left(1-\frac{3}{q^2a^2}\right)\tan qa + \frac{3}{qa}}{\left(1-\frac{3}{q^2a^2}\right)\tan qa + \frac{3}{qa}}.$$

This gives the law of decrease in the ellipticity of the strata in passing down from the surface to the centre.

By Art. 81, ϵ being now the ellipticity of the surface,

$$\frac{5}{3}a^2\phi(a)\left(\epsilon - \frac{1}{2}m\right) = \int_0^a \rho'\,\frac{d}{da'}\left(a'^5\epsilon'\right)da' = Q\int_0^a \frac{\sin qa'}{a'}\,\frac{d}{da'}\left(a'^5\epsilon'\right)da'$$

$$= Q\left\{a^4\epsilon\sin qa + \int_0^a a'^3\epsilon'\left(\sin qa' - qa'\cos qa'\right)da'\right\} \text{ by parts.}$$

Substituting for ϵ' from the ratio of ellipticities above, integrating and reducing, the integral in this expression

$$= \frac{\epsilon}{q^4}\,\frac{(\tan qa - qa)\sin qa}{\left(1-\frac{3}{q^2a^2}\right)\tan qa + \frac{3}{qa}}\left\{6q^2a^2 - 15 - \frac{q^3a^3 - 15qa}{\tan qa}\right\}.$$

$$\text{Also } \phi(a) = \frac{3Q}{q^2} \sin qa \left(1 - \frac{qa}{\tan qa}\right).$$

$$\text{Hence } \frac{m}{2\epsilon} =$$

$$1 - \frac{q^4a^4 - 3q^2a^2 + \frac{3q^3a^3}{\tan qa} + \left(1 - \frac{qa}{\tan qa}\right)\left(6q^2a^2 - 15 - \frac{q^3a^3 - 15qa}{\tan qa}\right)}{5\left(1 - \frac{qa}{\tan qa}\right)\left(q^2a^2 - 3 + \frac{3qa}{\tan qa}\right)}$$

$$= \frac{2q^2a^2 - q^4a^4 - \frac{q^3a^3}{\tan qa} - \frac{q^4a^4}{\tan^2 qa}}{5\left(1 - \frac{qa}{\tan qa}\right)\left(q^2a^2 - 3 + \frac{3qa}{\tan qa}\right)}.$$

Put $\frac{qa}{\tan qa} = 1 - z$ to facilitate the calculation;

$$\therefore \epsilon = \frac{5m}{2} \frac{z(q^2a^2 - 3z)}{-q^4a^4 + 3q^2a^2z - q^2a^2z^2} = \frac{5m}{2} \frac{1 - \frac{3z}{q^2a^2}}{3 - z - \frac{q^2a^2}{z}}.$$

When this is calculated for the surface, we shall be able to find the ellipticity of any stratum we like by the ratio of ellipticities found above.

106. To reduce this to numbers put, as before, $qa = 2{\cdot}4605$, $= 140^0\,58'\,35''$; then $\tan qa = -0{\cdot}8105$, $z = 4{\cdot}0359$,

$$q^2a^2 = 6{\cdot}0541,\ q^2a^2 \div z = 1{\cdot}5.$$

Hence the above formula for ϵ gives

$$\epsilon = \frac{5m}{2} \frac{1}{2{\cdot}5359} = \frac{m}{1{\cdot}01436} = \frac{1}{293}, \text{ since } m = \frac{1}{289}.$$

This is remarkably near the result obtained by geodesy.

107. Cor. 1. We will find the numerical value of the ellipticity of some of the strata, according to this law of density, with a view of showing at what rate the ellipticity diminishes in descending towards the centre. We will suppose

the mass divided into four shells and a central nucleus, the radius of the nucleus and the thickness of each shell being equal to one-fifth of the earth's radius. Taking the first value of qa found above, then for the outer surface and the four other surfaces we have

$$qa = 140^\circ 58' 35'',\ 112^\circ 46' 52'',\ 84^\circ 35' 9'',\ 56^\circ 23' 26'',\ 28^\circ 11' 43''$$
$$= 2{\cdot}4605,\ 1{\cdot}9684,\ 1{\cdot}4763,\ 0{\cdot}9842,\ 0{\cdot}4921 \text{ in arc};$$

$$\therefore \frac{qa}{\tan qa} = -3{\cdot}035906,\ -0{\cdot}826756,\ 0{\cdot}139920,\ 0{\cdot}654135,\ 0{\cdot}917944,$$

$$z = 4{\cdot}035906,\ 1{\cdot}826756,\ 0{\cdot}860080,\ 0{\cdot}345865,\ 0{\cdot}082056,$$

$$\frac{1}{z} = 0{\cdot}2478,\ 0{\cdot}5474,\ 1{\cdot}1627,\ 2{\cdot}8913,\ 12{\cdot}1868,$$

$$\frac{3}{q^2a^2} = 0{\cdot}4955,\ 0{\cdot}7743,\ 1{\cdot}3765,\ 3{\cdot}0971,\ 12{\cdot}3884.$$

$$\therefore \frac{1}{z} - \frac{3}{q^2a^2} = -0{\cdot}2477,\ -0{\cdot}2269,\ -0{\cdot}2138,\ -0{\cdot}2058,\ -0{\cdot}2016.$$

Hence by Art. 105

$$\frac{\text{ellipticity}}{\text{ellipticity of surface}} = 1,\ \frac{2269}{2477},\ \frac{2138}{2477},\ \frac{2058}{2477},\ \frac{2016}{2477}$$

$$= 1,\ \frac{1}{1{\cdot}09},\ \frac{1}{1{\cdot}16},\ \frac{1}{1{\cdot}20},\ \frac{1}{1{\cdot}23}.$$

108. Cor. 2. In a future Article we shall require to know the masses of the portions of the earth which lie within these surfaces to a first approximation, that is, neglecting their ellipticity. We will therefore calculate them here. Let E be the whole mass of the earth, and M the mass of a portion of it of which a is the mean radius. Then, if $A \sin qa \div a$ is the density,

$$M = \int_0^a 4\pi a^2 . A \frac{\sin qa}{a}\, da = 4\pi A \left(-\frac{a}{q} \cos qa + \frac{1}{q^2} \sin qa \right);$$

$$\therefore \frac{M}{E} = 1,\ 0{\cdot}6628,\ 0{\cdot}3371,\ 0{\cdot}1134,\ 0{\cdot}0153.$$

Hence the ratios of the masses of the four shells and of the nucleus to the whole mass are

$$0{\cdot}3372,\ 0{\cdot}3257,\ 0{\cdot}2237,\ 0{\cdot}0981,\ 0{\cdot}0153.$$

And the ratios of the mean densities of these to the mean density of the whole earth equal these multiplied by the fractions

$$\frac{5^3}{5^3-4^3},\ \frac{5^3}{4^3-3^3},\ \frac{5^3}{3^3-2^3},\ \frac{5^3}{2^3-1},\ \frac{5^3}{1^3};$$

and are, therefore,

$$0{\cdot}6910,\ 1{\cdot}1003,\ 1{\cdot}4717,\ 1{\cdot}7518,\ 1{\cdot}9125.$$

Fourth Test. **THE PRECESSION OF THE EQUINOXES.**

109. The last test we shall apply is the amount of Precession to which the fluid theory leads.

PROP. *To find the amount of Precession of the Equinoxes in the Earth, supposing its mass arranged according to the fluid law.*

110. The Annual Precession

$$= \frac{C-A}{C}\frac{3n'}{n}\cos I\left(1+\frac{n''^2}{n'^2}\frac{1-\frac{3}{2}\sin^2 i}{1+\nu}\right)180^0,$$

I = obliquity of the ecliptic = 23° 28′ 18″, i = inclination of Moon's orbit to ecliptic = 5° 8′ 50″, n and n' are the mean motions of the Earth round its axis and round the Sun, and their ratio = 365·26, n'' the mean motion of the Moon round the Earth = 27·32 days, ν = ratio of masses of Earth and Moon = 75. (See *Mechanical Philosophy*, Second Edition, Art. 470: also, changing the notation, *Airy's Tracts*, Fourth Edition, p. 213, Arts. 36, 38.) Substituting the above quantities,

$$\text{Annual Precession} = 16225''{\cdot}6\,\frac{C-A}{C},$$

where A and C are the principal moments of inertia of the mass, the latter about the axis of revolution. To find these let xyz be the co-ordinates to any element of the mass, $r\theta\omega$ be the polar co-ordinates to the same. Then the mass of this element $= -\rho r^2 d\mu d\omega dr$, $\mu = \cos\theta$. Also

$$\frac{y^2+z^2}{r^2} = 1-(1-\mu^2)\cos^2\omega = \frac{2}{3}+\left\{\frac{1}{3}-(1-\mu^2)\cos^2\omega\right\},$$

$$\frac{x^2+z^2}{r^2} = \frac{2}{3}+\left\{\frac{1}{3}-(1-\mu^2)\sin^2\omega\right\}, \quad \frac{x^2+y^2}{r^2} = \frac{2}{3}+\left(\frac{1}{3}-\mu^2\right).$$

The terms are here arranged as Laplace's Functions. (See Art. 39, Ex. 4.)

$$\therefore\ C-A = \int_{-1}^{1}\int_0^{2\pi}\int_0^r \rho\{(x^2+y^2)-(y^2+z^2)\}\, r^2 d\mu d\omega dr$$

$$= \int_{-1}^{1}\int_0^{2\pi}\int_0^r \rho r^4\left[\left(\frac{1}{3}-\mu^2\right)-\left\{\frac{1}{3}-(1-\mu^2)\cos^2\omega\right\}\right] d\mu d\omega dr.$$

Now r = radius of any stratum $= a\left\{1+\epsilon\left(\frac{1}{3}-\mu^2\right)\right\}$ (Art. 77);

$$\therefore \int_0^r \rho r^4 dr = \frac{1}{5}\int_0^a \rho\,\frac{d.r^5}{da}da$$

$$= \frac{1}{5}\int_0^a \rho\,\frac{d}{da}\left[a^5\left\{1+5\epsilon\left(\frac{1}{3}-\mu^2\right)\right\}\right] da$$

$$= \int_0^a \rho\left\{a^4+\frac{d.a^5\epsilon}{da}\left(\frac{1}{3}-\mu^2\right)\right\} da$$

$$= \sigma(a)+\psi(a)\left(\frac{1}{3}-\mu^2\right) \text{ suppose;}$$

$$\therefore\ C-A = \psi(a)\int_{-1}^{1}\int_0^{2\pi}\left(\frac{1}{3}-\mu^2\right)\{-\mu^2+(1-\mu^2)\cos^2\omega\}\, d\mu d\omega,$$

$$= \pi\psi(a)\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)(1-3\mu^2)\, d\mu = \frac{8\pi}{15}\psi(a), \text{ Art. 27.}$$

Also $C = \frac{8\pi}{3}\sigma(a)$, neglecting the small term $\psi(a)$.

Now $$\psi(a) = \int_0^a \rho \frac{d \cdot a^5 \epsilon}{da} da = \frac{5}{3} a^2 \phi(a) \left(\epsilon - \frac{m}{2}\right)$$

$$= \frac{5Qa^2}{q^2} \sin qa \left(\epsilon - \frac{m}{2}\right) z, \text{ by Arts. 81, 105.}$$

And putting $\rho = \frac{Q}{a} \sin qa$, and integrating by parts,

$$\sigma(a) = \int_0^a \rho a^4 da = Q \int_0^a a^3 \sin qa da$$

$$= Q\left(-\frac{a^3}{q} \cos qa + \frac{3a^2}{q^2} \sin qa + \frac{6a}{q^3} \cos qa - \frac{6}{q^4} \sin qa\right)$$

$$= \frac{Qa^2}{q^2} \sin qa \left\{3 - \frac{6}{q^2a^2} - \left(1 - \frac{6}{q^2a^2}\right) \frac{qa}{\tan qa}\right\}.$$

Hence substituting z, as before,

$$\frac{C-A}{C} = \frac{z}{2 + \left(1 - \frac{6}{q^2a^2}\right) z} \left(\epsilon - \frac{m}{2}\right).$$

Substituting for qa and z, the values in Art. 106,

$$\frac{C-A}{C} = 1{\cdot}9822 \left(\epsilon - \frac{1}{2} m\right);$$

$$\therefore \text{ Annual Precession} = 32162'' \left(\epsilon - \frac{1}{2} m\right).$$

But the Precession = 50″·1 by observation;

$$\therefore \epsilon - \frac{1}{2} m = 50{\cdot}1 \div 32162 = 0{\cdot}0015577;$$

$$\therefore \epsilon = 0{\cdot}0015577 + 0{\cdot}0017271 = \frac{1}{304}.$$

This is a satisfactory result, although it does not accord with the geodetic measure so closely as the direct calculation of the ellipticity in Art. 106 does. They both, however, bear testimony to the truth of the fluid theory and also in some measure of the law of density we have assumed.

111. The argument for the law of density hence deduced is, however, not very strong. For, as the strata are nearly spherical, almost any law might lead to right results by chosing the constants involved in the law rightly; especially as those results are the resultant effects of the whole mass, and not of the parts taken separately.

112. The Four Tests applied in the last twenty-four Articles give the following values of the ellipticity :—

$$\epsilon = \frac{1}{289}, \frac{1}{303}, \frac{1}{293}, \frac{1}{304}.$$

The average of these is $\frac{1}{297 \cdot 5}$.

§ 4. *The thickness of the Earth's Crust.*

113. In the next Chapter we shall test the accuracy of the theoretical results we have now arrived at, considering the earth as a fluid mass, by an appeal to actual measurements. That it is not wholly fluid now, the existence of continents and ocean-beds attests. It is a question how far down this solidity prevails. It is supposed by some that the crust may be of comparatively small thickness, so thin as to be influenced in its position of elevation and depression by the fluid mass below, on which it is in fact imagined to float.

Mr Hopkins has endeavoured to ascertain how far the interior of the Earth may at present be fluid, by calculating the value of the Precession upon the supposition of the mass being a spheroidal shell of heterogeneous matter, enclosing a heterogeneous fluid mass, consisting of strata increasing according to the law we have used. In three memoirs in the *Philosophical Transactions* of 1839, 1840, and 1842, he enters upon a complete investigation of this subject. We will give the evidence upon which he rests his conclusion that the crust is very thick.

PROP. *To trace the argument drawn from Precession to show that the crust is of considerable thickness.*

114. Mr Hopkins has deduced the following formula (in

which we have changed the notation to suit the present treatise),

$$\frac{P-P'}{P} = 1 - \frac{\epsilon'}{\epsilon} - \frac{\int_a^a \rho' \frac{d.a'^5(\epsilon''-\epsilon')}{da'}\, da'}{2\epsilon a^3 \int_0^a \rho' a' da' + \epsilon \int_a^a \rho' \frac{d.a'^5}{da'}\, da'},$$

where P is the precession of the equinoxes of a homogeneous spheroid of ellipticity ϵ, which by calculation $= 57''$ nearly if $\epsilon = \frac{1}{300}$; P' is the precession of the heterogeneous shell, the outer and inner ellipticities being ϵ and ϵ': this $= 50''{\cdot}1$ by observation.

The success of the calculation depends upon a remarkable result at which he has arrived, that the precession caused by the disturbing forces in a homogeneous shell filled with homogeneous fluid, in which the ellipticities of the inner and outer surfaces are the same, is the same whatever the thickness of the shell. It is therefore the same for a spheroid solid to the centre. The formula above given is the relation of the amounts of precession in two shells, one heterogeneous and the other homogeneous; and, as the thickness is the quantity sought, neither of these amounts could be calculated, and therefore the relation expressed in the above formula would be of no avail. But in consequence of the property that the precession of the shell, when it and the fluid are homogeneous, is the same as that of the spheroid, this difficulty is overcome; and P can be calculated without knowing the thickness, and therefore P' will be known.

We have shown (Art. 82) that the strata decrease in ellipticity in passing downwards: hence $\epsilon'' - \epsilon'$ is never negative, and the fraction on the right hand in the above formula is never negative, and is never so large as unity: let it $= \beta$. Hence

$$\frac{\epsilon'}{\epsilon} = \frac{7}{8} - \beta, \text{ or } \epsilon' \text{ is less than } \frac{7}{8}\epsilon;$$

and therefore, because the ellipticity decreases in descending, the thickness must be greater than would correspond with an

ellipticity of the inner surface of the shell equal to 7-8ths of that of the outer surface.

If solidification took place solely from pressure, the surfaces of equal density would be surfaces of equal degrees of solidity. If we use the formula for finding ϵ in Art. 105, and make $qa = 150^0$, and the mean density $= 2{\cdot}4225$ times the superficial density (from Mr Airy's Harton calculation), then if $\epsilon' = \frac{7}{8}\epsilon$ in the formula of Art. 105, we have, after reduction, $a = \frac{3}{4}$ a, or the thickness equal to one fourth of the radius, or 1000 miles. If a smaller ratio of densities is used than 2·4225, the thickness is greater. (Mr Hopkins shows also that a ratio a little larger than 3 makes the thickness 1-5th of the radius: but this ratio is too large. The ratio generally used is about 2·2).

But solidification depends upon temperature, as well as upon pressure. In his third memoir (*Phil. Trans.* 1842), Mr Hopkins shows that the isothermal surfaces increase in ellipticity in passing downwards. If temperature alone regulated the solidification, these surfaces would be the surfaces of equal solidity. But since both pressure and temperature have their effects, the ellipticities of the surfaces of equal solidity must lie between those of the isothermal and the equi-dense surfaces. Hence the surface of equal solidity at any depth will be more elliptic than the surface of equal density at that depth: and therefore the inner surface of the solid shell, of which the ellipticity is $\frac{7}{8}\epsilon$, must be at a depth corresponding to a stratum of equal density of smaller ellipticity than $\frac{7}{8}\epsilon$, that is, at a greater depth than 1000 miles.

In the above reasoning β has been neglected. If its value be used, it strengthens the argument for a greater thickness than 1000 miles.

We may, therefore, safely conclude that 1000 miles is the least thickness of the solid crust. In the calculation it has been assumed that the transition from the solid shell to the fluid nucleus is abrupt. This will hardly be the case. The

above result will therefore apply to the *effective* surface, lying near the really solid shell. But in consequence of the tendency, as shown above, of every cause being to prove that the crust is really thicker than 1000 miles, we may safely take this to be its least limit*.

115. Professors Hennessy, Haughton and W. Thomson have written upon this subject: see *Phil. Trans.* 1851, *Transactions of the Royal Irish Academy*, 1852, and *Phil. Trans.* 1862. The first makes the thickness lie between 18 and 600 miles. But in his calculation he assumes that the shell is so rigid as to resist, without change of form, the internal pressure which arises from the inner surface ceasing to be one of fluid equilibrium: an assumption which cannot be considered admissible. Moreover he supposes that in cooling the outer shell will contract less than the fluid nucleus; which can hardly be true.

Mr Haughton's investigation is simply a problem of densities, and determines nothing whatever regarding the ratio of the solid to the fluid parts of the Earth. (See *Philosophical Magazine*, Sept. 1860.)

Mr Thomson, in his paper on the "Rigidity of the Earth," confirms Mr Hopkins' result, as will appear from the investigation with which this Chapter closes.

116. In the last edition we gave an argument in favour of a great thickness, arising from the tendency of the weight of the enormous mass of the Himmalaya mountains to break down the crust if it is thin; and of the fluid mass below extensive ocean-beds to burst them up under the same circumstances. Subsequent calculations, however, on another subject have shown that the crust below the mountains must be of less density, and that below ocean-beds of greater density than the average, so as to produce a very considerable compensation. If this is the case the downward pressure in the first instance

* It will be observed that the ellipticity of the surface of the earth is in these pages always denoted by the symbol ϵ, the mean radius of the surface being a. Owing, however, to the want of variety of Greek type this same symbol has occasionally been used to denote the ellipticity of an internal stratum of mean radius a. No confusion will arise from this, as the meaning of the symbol in each case is sufficiently obvious.

and the upward pressure in the second might be too feeble to produce fracture even if the crust be thin. (See Arts. 132—135.)

117. We will conclude this Chapter by giving some account of Professor W. Thomson's remarkable investigation regarding the "Rigidity of the Earth," in the first place making a calculation which will be of use. He accepts the result of Mr Hopkins' calculation, and indeed thinks that it might have been pressed further. His own investigations, an abstract of which we now give, point out that the earth's mass must possess such a degree of rigidity as to be altogether inconsistent with a crust of moderate thickness.

PROP. *A mass in the form of a prolate spheroid of small ellipticity consists partly of a nucleus of spherical shells concentric with the spheroid, the remaining portion of the spheroid being homogeneous: to find the form of the level surface, or surface at every point of which the resultant attraction of the whole mass acts in the normal, passing through the equator of the spheroid.*

118. Let a and $c = a(1+\epsilon)$ be the semi-axes, suppose a sphere of radius a to be inscribed touching the spheroid in its equator: ρ the density of the outer part of the spheroid, $N.\rho$ the mean density of the mass within the inscribed sphere. We may conceive the mass made up of a homogeneous prolate spheroid of density ρ, and a concentric sphere of radius a and mean density $(N-1)\rho$. Then by Art. 14, the attractions of the homogeneous spheroid parallel to the axis on a point xyz are

$$\frac{4}{3}\pi\rho\left(1+\frac{2}{5}\epsilon\right)x,\quad \frac{4}{3}\pi\rho\left(1+\frac{2}{5}\epsilon\right)y,\quad \frac{4}{3}\pi\rho\left(1-\frac{4}{5}\epsilon\right)z,$$

and the attractions of the sphere

$$\frac{4}{3}\pi\rho(N-1)\frac{a^3}{r^3}x,\quad \frac{4}{3}\pi\rho(N-1)\frac{a^3}{r^3}y,\quad \frac{4}{3}\pi\rho(N-1)\frac{a^3}{r^3}z.$$

If X, Y, Z are the total attractions parallel to the axes at the point xyz, then, that their resultant at that point may act in the normal, we must have

$$Xdx + Ydy + Zdz = 0,$$

which, by substitution integration and division, gives

$$\left(1+\frac{2}{5}\epsilon\right)(x^2+y^2)+\left(1-\frac{4}{5}\epsilon\right)z^2-2\,(N-1)\,\frac{a^3}{r}=\text{const.},$$

$$\text{or}\ \left(1+\frac{2}{5}\epsilon-\frac{6}{5}\epsilon\sin^2 l\right)r^2-2\,(N-1)\,\frac{a^3}{r}=\text{const.}\ (l=\text{latitude}).$$

Suppose $r=a\,(1+\epsilon'\sin^2 l)$, which can evidently be made to satisfy this equation, is the equation to the level surface through the equator. By substituting for r and equating the constant parts and also the parts depending on $\sin^2 l$, the latter leads to

$$2\epsilon'-\frac{6}{5}\epsilon+2\,(N-1)\,\epsilon'=0,$$

$$\therefore\ \epsilon'=\frac{3}{5N}\,\epsilon,$$

which gives the form of the level surface.

PROP. *To explain how the Tides can be used as a test of the degree of Rigidity of the Earth's mass.*

119. It is necessary to premise, that, in an elaborate and difficult investigation on the deformation of a spheroidal elastic mass acted upon by external forces, Professor Thomson has deduced the following formula (*Phil. Trans.* 1863, p. 574),

$$h=\frac{h'}{1+\frac{19}{2}\,\frac{n}{gwr}},$$

where h' and h are the polar elevations of a prolate spheroid drawn out from a spherical figure by an external force (resembling the combined action of the sun and moon in raising the tidal wave), the spheroid being considered a homogeneous mass of incompressible *fluid* in the first instance, and of incompressible *solid* matter in the second: in both cases the total mass is equal to the mass of the earth; w denotes the mass of a unit of volume, that is, the density; r the

radius of the globe, and n the rigidity of the solid substance, that is, $n \cdot k$ is the force necessary to draw a unit of section of the solid through an extremely small space k.

Now suppose the earth an absolutely rigid sphere, and H the polar elevation of the prolate spheroid, or level surface, which is everywhere perpendicular to the resultant action of the sun and moon, which in the actual ocean, produces the tidal wave. Then H is the height of high above low water, in the equilibrium theory of the tides, of an ocean of infinitely small density covering a rigid earth.

Suppose instead of the above that the earth is covered with an ocean, the earth still being a perfectly rigid sphere, and its mean density N times that of the ocean. Let H' be the polar elevation of the prolate spheroid into which the sun and moon draw the ocean. Then by Art. 118, the terrestrial gravitation level would be disturbed by this cause, from the spherical surface to a spheroidal surface of which the polar elevation is $(3 \div 5N)\, H'$, by the attraction of the ocean in its altered form. The polar elevation in the level surface, before noticed as produced by the direct action of the sun and moon, must be added to this to give the polar elevation of the actual equilibrium level of the free surface of the ocean. Hence

$$H' = H + \frac{3}{5N} H', \text{ or } H' = \frac{H}{1 - \frac{3}{5N}}.$$

It will be observed that h' and H' are similar quantities, but for oceans of different density: when $N = 1$, $h' = H'$, and this $= \frac{5}{2} H$, and

$$\therefore\ h = \frac{5}{2} \frac{H}{1 + \frac{19}{2} \frac{n}{wgr}}.$$

We have thus far been considering the earth to be absolutely rigid. If its solid mass is drawn up from a spherical form to a polar elevation h by the sun and moon, the attraction of the protuberant mass will change the gravitation level from a sphere to a prolate spheroid of polar elevation

$(3 \div 5)\, h$: and this as before should be added to H to find the whole effect of the sun and moon in changing the gravitation level. The sum will be the absolute tidal elevation above the sphere, of an ocean of infinitely small density covering the elastic globe. By subtracting h, the tidal elevation of the solid itself, we have the difference between high and low water

$$= H - \frac{2}{5} h, \quad = \frac{\frac{19}{2} \frac{n}{gwr} H}{1 + \frac{19}{2} \frac{n}{gwr}}.$$

If the earth be perfectly rigid, n is infinite, and this expression becomes H, as it ought to do. For iron or steel

$$n = 501 \times 10^8,$$

the unit of mass being 1 lb., the unit of space 1 foot. This makes the above expression for the height of the tide equal to $0{\cdot}59H$ or about $\frac{3}{5} H$. For glass

$$n = 2160000 \times 144 \times 32{\cdot}2,$$

which makes the tide $\frac{2}{9} H$.

"Imperfect," Professor Thomson remarks, "as the comparison between theory and observation as to the absolute height of the tides has been hitherto, it is scarcely possible to believe that the height is in reality only two-ninths of what it would be if, as has hitherto been universally assumed in tidal investigations, the earth were perfectly rigid. It seems therefore nearly certain, with no other evidence than is afforded by the tides, that the tidal effective rigidity of the earth must be greater than that of glass."

PROP. *To explain how Precession can be used as a test of the Earth's Rigidity.*

120. Conceive the earth to be a fluid mass revolving in a

day about its axis, and drawn by centrifugal force into a spheroidal figure having the same amount of protuberant matter as the earth actually has. Suppose also that the combined action of the sun and moon produces a tidal wave on each side of the earth which is superimposed upon the spheroid of revolution. As the earth in this case would have no rigidity whatever, it would therefore have no precession. Conceive things to remain the same, except that the fluid becomes an elastic solid, yielding as the fluid to the varying influence of the sun and moon, so as to produce a tidal wave superinduced as before on the spheroid of revolution. This mass would still have no precession. As the mass is now solid, though elastic, the sun and moon, by attracting the nearer equatorial parts more, and the further equatorial parts less, than the centre, would have a tendency to cause it to rotate round an axis in the equator and produce precession. But as, in the case supposed, no precession takes place, this tendency of the sun and moon to produce precession must be exactly counterbalanced by some opposite tendency: that tendency is the effect of the centrifugal force on the protuberant parts of the tidal wave drawn up on the solid by the sun and moon. This effect, therefore, of the solid tidal wave would in amount exactly equal the actual precessional motion of the earth, on the hypothesis of the earth's mass being perfectly rigid, though it would act in the opposite direction.

Now by the last Article it appears, that if the earth have the rigidity of steel or glass the tides would be reduced to 3-5ths or 2-9ths of their amount on the supposition that the earth's rigidity is perfect; that is, the deformation of the solid parts beneath the ocean would be 2-5ths or 7-9ths of that amount. The result would be that 2-5ths or 7-9ths of the precession caused in a rigid earth would be balanced if the mass have only the rigidity of steel or glass. As no such effect is detected by observation it must be presumed that the rigidity of the earth is decidedly very great.

"The close agreement," Professor Thomson remarks, "between the observed amounts of precession and nutation and the results of theory on the hypothesis of perfect rigidity, renders it impossible to believe that there is enough of

elastic yielding to influence the phenomena to any considerable extent. It is worthy of remark, however, that in general the theoretical estimates of the amount of precession have been somewhat above the true amount demonstrated by observation. It seems not altogether improbable that this discrepancy is genuine, and is to be explained by some small amount of deformation experienced by the solid parts of the earth, under lunar and solar influence."

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CHAPTER II.

THE FIGURE OF THE EARTH, DETERMINED BY GEODETIC OPERATIONS.

121. In the previous Chapter we have shown that if the Earth be considered a fluid mass the form of the surface will be an oblate spheroid of small ellipticity, its axis coinciding with the axis of revolution, and the surface being everywhere at right angles to the direction of gravity; and further, that upon assuming that the density of the strata varies according to a certain probable law, the ellipticity is somewhat greater than $\frac{1}{300}$.

We propose to submit this to the test of measurement, by enquiring whether an ellipse can be found with its axis coinciding with the axis of the Earth and cutting the plumb-line at stations along it at right angles; and whether the ellipticity of that ellipse is about $\frac{1}{300}$.

The method of doing this is as follows. A base-line, about 5 or 6 miles in length, is measured with extreme accuracy, near the meridian, the curvature of which we are to find. By a series of triangles this base is connected with a number of stations in succession lying near the meridian, the angles and sides of which are calculated or observed, as the case may be. Thus a connexion is established between the original base and a second base at the termination of the chain of triangles, and the length of this second base obtained by calculation. It is then measured, as the first was, and by a comparison of the calculated and measured results the correctness or not of

the operations is tested. This having been satisfactorily performed, the projections of the sides of the triangles on the meridian are found, and their sum gives the length of the meridian arc between its two extremities. The latitudes of these extremities are then observed with great care, and from these data the form of the ellipse, of which the arc is a part, is found by the principles of conic sections, as we shall now show.

It is obvious that the actual surface of the earth is of a very irregular form, being diversified by mountains and valleys. In our investigation, at any rate in the first instance, these are not taken account of; the whole is supposed to be levelled down and all the measures which are taken are reduced down to the sea-level, the sea being supposed to have the spheroidal form, since it is a free surface. The sea-level at any place means the level at which sea-water would stand if let in from the sea by a canal.

§ 1. *The determination of the Mean Figure of the Earth.*

PROP. *To find the length of an arc of meridian in terms of the amplitude, the semi-axis major, the ellipticity (the ellipticity being small), and the middle latitude.*

122. Let l and l' be the latitudes of the extremities of the arc, m the mean of these or the middle latitude; λ the amplitude of the arc or the difference between the latitudes; a, b, and ϵ the semi-axes and ellipticity; s the length of the arc, r the radius vector, and θ the angle r makes with the major axis. Then

$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}, \quad \tan l = \frac{a^2}{b^2}\tan\theta;$$

$$\therefore \frac{1}{r^2} = \frac{a^2\cos^2 l + b^2\sin^2 l}{a^4\cos^2 l + b^4\sin^2 l}, \text{ putting } b = a(1-\epsilon),$$

$$r = a(1 - \epsilon\sin^2 l), \text{ neglecting } \epsilon^2 \ldots$$

$$\frac{dr}{dl} = -2a\epsilon\sin l\cos l, \quad \frac{d\theta}{dl} = 1 - 2\epsilon + 4\epsilon\sin^2 l;$$

$$\therefore \frac{ds}{dl} = \sqrt{r^2 \frac{d\theta^2}{dl^2} + \frac{dr^2}{dl^2}} = a\,(1 - 2\epsilon + 3\epsilon \sin^2 l)$$

$$= a\left(1 - \frac{1}{2}\epsilon - \frac{3}{2}\epsilon \cos 2l\right);$$

$$\therefore s = a\left\{\left(1 - \frac{1}{2}\epsilon\right)(l - l') - \frac{3}{4}\epsilon\,(\sin 2l - \sin 2l')\right\}$$

$$= \frac{1}{2}(a+b)\,\lambda - \frac{3}{2}(a-b)\sin\lambda \cos 2m.$$

123. Cor. 1. If λ be small, not exceeding 12^0, we may put $\sin\lambda = \lambda$ in this formula; then

$$\frac{s}{\lambda} = \frac{a+b}{2} - 3\,\frac{a-b}{2}\cos 2m.$$

Cor. 2. The value of λ in terms of s including the square of the ellipticity is given by the formula, which may easily be deduced from the last Article, viz.:

$$\lambda = \frac{s}{a}\left[1 + \epsilon\left(\frac{1}{2} + \frac{3}{2}\frac{\sin\lambda}{\lambda}\cos 2m\right) + \epsilon^2\left\{\left(\frac{1}{2} + \frac{1}{3}\frac{\sin\lambda}{\lambda}\cos 2m\right)^2 + \frac{3}{16} + \frac{3}{4}\frac{\sin\lambda}{\lambda}\cos 2m - \frac{15}{32}\frac{\sin\lambda}{\lambda}\cos\lambda\cos 4m\right\}\right].$$

Cor. 3. Let S be the length of an arc of longitude in latitude l, L the longitudinal amplitude of the arc. Then the radius of the circle of longitude

$$= r\cos\theta = a\cos l\,(1 + \epsilon\sin^2 l) = \cos l\,\{a + (a-b)\sin^2 l\};$$

$$\therefore S = L\cos l\,\{a + (a-b)\sin^2 l\}.$$

Prop. *To obtain formulæ for finding the semi-axes and ellipticity, when the lengths, amplitudes, and middle latitudes of two small arcs are known; and to ascertain what arcs are adapted to give the best results.*

124. Let $s\lambda m$, $s'\lambda' m'$ be the lengths, amplitudes, and middle latitudes;

$$\therefore \frac{s}{\lambda} = \frac{a+b}{2} - 3\frac{a-b}{2}\cos 2m, \quad \frac{s'}{\lambda'} = \frac{a+b}{2} - 3\frac{a-b}{2}\cos 2m';$$

$$\therefore \frac{a-b}{2} = \frac{1}{3}\frac{\frac{s}{\lambda} - \frac{s'}{\lambda'}}{\cos 2m' - \cos 2m}, \quad \frac{a+b}{2} = \frac{\frac{s}{\lambda}\cos 2m' - \frac{s'}{\lambda'}\cos 2m}{\cos 2m' - \cos 2m},$$

by which a and b and therefore ϵ are found.

The effect on the axes of any error in the amplitudes will be found by differentiating the above formulæ. In the denominators of the resulting expressions the quantity

$$\cos 2m - \cos 2m'$$

will appear. The errors in the axes consequent on errors in the observed amplitudes will, therefore, be least when this quantity is a maximum. Suppose one arc is chosen in the southern half of the quadrant, $\cos 2m$ is positive; then

$$2m' = 180^\circ \text{ or } m' = 90^\circ$$

will give the best result. Suppose one arc is in the northern half, $\cos 2m$ is negative; then $2m' = 0$ will give the best result. Hence the nearer one arc is to the pole and the other to the equator, the less will errors in the data affect the calculated form of the ellipse. This will be illustrated in the following examples. The data are taken from the Volume of the *British Ordnance Survey*, pp. 743, 757.

125. Ex. 1. Compare the two parts of the English Arc, from Saxaford (60° 49′ 39″) to Clifton (53° 27′ 30″), measuring 2692754 feet, and from Clifton to Southampton (50° 54′ 47″), measuring 928774 feet.

$$\lambda = 7^\circ\,22'\,9'' = 26529'', \quad \lambda' = 2^\circ\,32'\,43'' = 9163'',$$

$$2m = 114^\circ\,17'\,9'', \quad 2m' = 104^\circ\,22'\,17'',$$

$$\therefore \frac{1}{2}(a-b) = 59419, \quad \frac{1}{2}(a+b) = 20863630.$$

$$a = 20923049, \quad b = 20804211, \quad \epsilon = \frac{1}{176} = \frac{1+0{\cdot}70}{300}.$$

Ex. 2. Compare the two parts of the Indian Arc from Kaliana (lat. 29° 30′ 48″) to Kalianpur (24° 7′ 11″), the length being 1961138 feet, and that between Kalianpur and Damargida (18° 3′ 15″), the length being 2202905 feet.

$$\lambda = 5^\circ\,23'\,37'' = 19417'', \quad \lambda' = 6^\circ\,3'\,56'' = 21836'',$$

$$2m = 53^\circ\,37'\,59'', \quad 2m' = 42^\circ\,10'\,26'',$$

$$\therefore \frac{1}{2}(a-b) = 54064, \quad \frac{1}{2}(a+b) = 20929075 \text{ feet},$$

$$a = 20983139, \quad b = 20875011, \quad \epsilon = \frac{1}{194} = \frac{1+0{\cdot}55}{300}.$$

Ex. 3. Compare the arc between Kalianpur and Damargida with that between Damargida and Punnæ (8° 9′ 31″) the length being 3591784 feet.

$$\lambda = 6^\circ\,3'\,56'' = 21836'', \quad \lambda' = 9^\circ\,53'\,44'' = 35624'',$$

$$2m = 42^\circ\,10'\,26'', \quad 2m' = 26^\circ\,12'\,46''$$

$$\therefore \frac{1}{2}(a-b) = 26194, \quad \frac{1}{2}(a+b) = 20867130 \text{ feet},$$

$$a = 20893324, \quad b = 20840936, \quad \epsilon = \frac{1}{399} = \frac{1-0{\cdot}25}{300}.$$

Ex. 4. Compare the arcs between Kaliana and Kalianpur and between Damargida and Punnæ.

$$\lambda = 5^\circ\,23'\,37'' = 19417'', \quad \lambda' = 9^\circ\,53'\,44'' = 35624'',$$

$$2m = 53^\circ\,37'\,59'', \quad 2m' = 26^\circ\,12'\,46'';$$

$$\therefore \frac{1}{2}(a-b) = 39867, \quad \frac{1}{2}(a+b) = 20903830,$$

$$a = 20943697, \quad b = 20863963, \quad \epsilon = \frac{1}{262} = \frac{1+0{\cdot}15}{300}.$$

Ex. 5. Compare the arcs between Damargida and Punnæ and between Clifton and Southampton.

$$\lambda = 9^\circ\,53'\,44'' = 35624'', \quad \lambda' = 2^\circ\,32'\,43'' = 9163'',$$

$$2m = 26^\circ\,12'\,46'', \quad 2m' = 104^\circ\,22'\,17'';$$

$$\therefore \frac{1}{2}(a-b) = 32208, \quad \frac{1}{2}(a+b) = 20883305,$$

$$a = 20915513, \quad b = 20851097, \quad \epsilon = \frac{1}{325} = \frac{1-0{\cdot}08}{300}.$$

It will be seen in these successive examples that the ellipticity is nearer and nearer to that deduced from the fluid theory; when the arcs compared are near each other the resulting ellipticity differs much from that value; but when they are more distant from each other, as in the fifth example, the result is far more accordant. This agrees with what was deduced from the formulæ in the last Article. If there were no errors in the data, viz. in the observed amplitudes and measured arcs, the results ought to come out in complete accordance with each other, if the figure of the Earth be truly spheroidal; for the formulæ are sufficiently exact for this purpose.

PROP. *To explain the cause of the ellipses, determined from the several pairs of arcs, differing from each other.*

126. We have assumed, (1) that the meridian arc is an ellipse, that being the form which it would have were the Earth fluid: (2) that the plumb-line at all stations of the meridian is a normal to this ellipse. These suggest in what direction we are to look for an explanation of the discrepancies in the results.

First. It is obvious that the form of equilibrium no longer actually exists, as all the variety of hill and dale, mountain and table-land and ocean-surface, sufficiently testifies. Geology teaches the same more generally and philosophically. Extensive portions now dry land were once at the bottom of the ocean, receiving the fossil deposits and burying them in the detritus of rocks, which time wore down, to become, as they are now, the records of their own history. Changes of level must therefore have taken place on a large scale. Landmarks in Scandinavia, the temple of Serapis at Puzzuoli, the ancient and recent coral-reefs in the Pacific, as pointed out

by Mr Darwin, all testify that these changes of level are still slowly going on. It has been suggested, with great probability, that it is caused by the expansion and contraction of vast portions of rock in the interior of the Earth arising from variations in temperature produced by chemical changes. Whatever the cause, the fact is certain. The Earth's form can no longer be a form of fluid-equilibrium, although the average form may be so.

Secondly. The plumb-line may not in all cases be perpendicular to the mean ellipse. Local attraction is sufficient to produce material errors in the vertical, and therefore in the amplitudes determined by meridian zenith distances of stars. For instance (Art. 55, Ex. 2), an error as great as 5″ was discovered at Takal K'hera in Central India by Colonel Everest, arising from the attraction of a distant table-land. Sir Henry James has shown that a deflection of about the same amount occurs at Arthur's Seat, Edinburgh (*Phil. Trans.* 1857). We have mentioned that the attraction of the Himmalaya Mountains produces a deflection amounting to as much as 28″ at the northern extremity of the Great Indian Arc (Art. 62, Ex. 1). We have calculated elsewhere (see Art. 62, Ex. 2, and *Phil. Trans.* for 1859) that the deficiency of matter in the vast ocean south of India causes such deflections as 6″, 9″, 10″·5, 19″·7 at various stations: and (Art. 64) we have shown that it is not improbable that extensive but slight variations of density prevail in the interior of the Earth, the causes of which are not visible to us as mountain masses and vast oceans are, sufficient to produce errors in the plumb-line quite as great as and even greater than most of those already enumerated. These seem abundantly to account for the variety in the calculated semi-axes and ellipticities in the last Article, derived as they are from uncorrected observations.

127. Mr Airy has entered very thoroughly into a comparison (see *Figure of the Earth, Encyc. Metrop.*) of the various arcs measured in different parts of the world. He has used them according to their importance and value, as determined by the circumstances under which they were measured and observed.

128. The late M. Bessel devised a method by which the

results of all the surveys in different parts of the world might be brought to bear simultaneously upon the problem. This method is followed by Captain A. Clarke, R.E. in his Chapter on the figure of the earth at the end of the *British Ordnance Survey* Volume. The arcs which he uses in his calculation for determining the mean figure of the earth are eight in number; viz. the Anglo-Gallic, Russian, Indian II (or Great Arc), Indian I, Prussian, Peruvian, Hanoverian, and Danish arcs. These consist of fifty-eight subordinate divisions, the lengths of which have been measured and the latitudes of their extremities described. The method which Bessel invented was this: corrections, expressed in algebraical terms, are applied to the latitudes of the several stations dividing the arcs into their subordinate parts, such as to make their measured lengths exactly fit an ellipse. The values of the axes of this ellipse are then so determined as to make the sum of the squares of these corrections a minimum: *that* is the ellipse which most nearly represents the observations and measures, and is therefore taken to be the mean ellipse.

PROP. *To obtain a formula for correcting the amplitude of an arc, so as to make its measured length accord with a given ellipse.*

129. The length of an arc is, by Art. 122,

$$s = \frac{1}{2}(a+b)\lambda - \frac{3}{2}(a-b)\sin\lambda\cos 2m.$$

Suppose now that xx' are small corrections which must be applied to the observed latitudes to make the measured arc fit the ellipse of which a and b are the semi-axes; then λ and m, being obtained from observation, will not, when substituted in the above formula, give the measured value of s; but

$$\lambda + x' - x \text{ and } 2m + x' + x$$

must be substituted instead of them. Hence omitting very small quantities,

$$s = \frac{1}{2}(a+b)(\lambda + x' - x) - \frac{3}{2}(a-b)\sin(\lambda + x' - x)\cos 2m$$

$$= \frac{1}{2}(a+b)\lambda - \frac{3}{2}(a-b)\sin\lambda\cos 2m$$

$$+ \frac{1}{2}(x'-x)\{(a+b)\lambda - 3(a-b)\cos\lambda\cos 2m\};$$

$$\therefore\ x' - x = \frac{2s - (a+b)\lambda + 3(a-b)\sin\lambda\cos 2m}{(a+b)\lambda - 3(a-b)\cos\lambda\cos 2m}.$$

Now the mean radius of the earth is known not to differ much from 20890000 feet, and the ellipticity from $\frac{1}{300}$. It is therefore convenient to put a and b under the form

$$\frac{a+b}{2} = \left(1 - \frac{u}{10000}\right)20890000,$$

$$\frac{a-b}{2} = \frac{1}{600}\left(1 - \frac{u}{10000} + \frac{v}{50}\right)20890000;$$

$$\text{and } \therefore\ \epsilon = \frac{1}{300}\left(1 + \frac{v}{50}\right);$$

where the squares of u and v may be neglected. When these are substituted in the formula it may be put in the following form,

$$x' = m + \alpha u + \beta v + x,$$

where m, α, β are functions of the observed latitudes and the measured length and other numerical quantities only.

The values of m, α, β have been calculated in the *Ordnance Survey* Volume for the 85 divisions of the 8 arcs mentioned in Art. 128.

130. In pursuing the process described in Art. 128, the ten quantities u, v, x_1, $x_2 \ldots x_8$ are all considered as variables, to be determined so as to make the sum of the squares of the corrections a minimum. The result is, that $u = -0{\cdot}3856$,

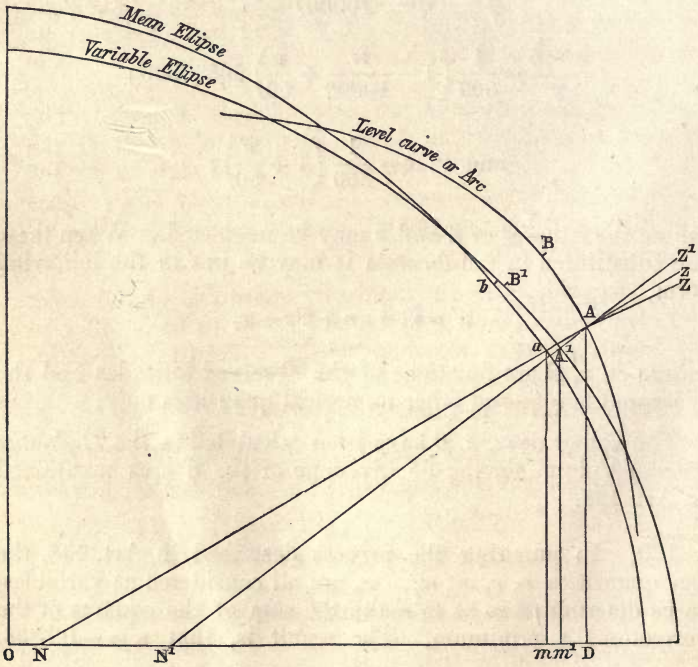
$v = 1{\cdot}0620$; and these make the semi-axes and ellipticity of the mean ellipse as follows:

$$a = 20926348,\ b = 20855233,\ \epsilon = \frac{1}{294}.$$

But this process, we think, is not correct. Although $x_1 \ldots x_8$ are unknown quantities, yet they are not variables independent of u and v. This we shall show in the following Proposition.

PROP. *To determine the correction of the latitude of the reference-station of an arc, in terms of the axes of the variable ellipse and the deflection of the plumb-line at the station arising from local attraction.*

131. In the accompanying diagram the plane of the paper



is the plane of the meridian in which the arc, of which AB is one section, has been geodetically measured. A is the reference-station of the several portions of the whole arc. AZ is the vertical at A in which the plumb-line hangs. The two curves, of which $A'B'$ and ab are portions, are a variable ellipse and the mean ellipse having the same centre O and their axes in the same lines, the mean ellipse being what the variable ellipse becomes when the values are substituted for u and v which make the sum of the squares of the errors a minimum: $Z'AA'N'$ and $zAaN$ are normals through A to these two ellipses; AD, $A'm'$, am are perpendicular to OD.

Now, if the earth had its mean form, a plumb-line at A would hang in the normal zA to the mean ellipse; but it hangs actually in ZA. Hence ZAz is the deflection (northward in the diagram) which the plumb-line suffers from the local attraction arising from the derangement of the figure and mass of the earth from the mean. This angle is some constant but unknown quantity t, t being reckoned positive when the deflection is northward. This quantity t is part of the correction ZAZ', or x, added to the observed latitude of A before applying the principle of least squares. The other part is zAZ', which I will now calculate: it is the angle between the two normals drawn through A to the variable and the mean ellipses. By the property of an ellipse of which the ellipticity is small,

$$ON = 2\epsilon \,.\, Om, \text{ and } ON' = 2\epsilon' \,.\, Om'.$$

Also as Om, Om', OD differ only by quantities of the order of the ellipticities, they may be put equal to each other in small terms, because we neglect the square of the ellipticities.

$$\therefore \angle zAZ' = \angle NAN' = \angle AN'D - \angle AND$$

$$= \tan^{-1} \frac{\cot AND - \cot AN'D}{1 + \cot AND' \cot AND} = \tan^{-1} \frac{(ND - N'D)\, AD}{AD^2 + ND \,.\, N'D}$$

$$= \tan^{-1} \frac{(ON' - ON)\, AD}{AD^2 + DO^2} = \tan^{-1} \frac{2\,(\epsilon' - \epsilon)\; OD \,.\, AD}{AD^2 + DO^2}$$

$$= \tan^{-1} (\epsilon' - \epsilon) \sin 2l = (\epsilon' - \epsilon) \sin 2l \frac{1''}{\sin 1''},$$

l being the observed latitude of A.

Suppose that v and V are the values of v for the variable and the mean ellipses. Then by the value of ϵ in Art. 129

$$\angle zAZ' = \frac{\sin 2l}{15000 \sin 1''} (v - V) = 13''{\cdot}75 \sin 2l (v - V)$$

$$= n (v - V) \text{ suppose } \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$\therefore x = t + n (v - V).$$

PROP. *To obtain formulæ for calculating the Mean Figure of the Earth, taking into account local attraction.*

132. If we adopt Bessel's method with the necessary correction pointed out above, the sum of the squares of errors, which is to be differentiated with respect to u and v to obtain a minimum, is

$$\{n_1 (v - V) + t_1\}^2 + \{m_1 + \alpha_1 u + \beta_1 v + n_1 (v - V) + t_1\}^2$$
$$+ \{m'_1 + \alpha'_1 u + \beta'_1 v + n_1 (v - V) + t_1\}^2 + \ldots$$
$$\{n_2 (v - V) + t_2\}^2 + \{m_2 + \alpha_2 u + \beta_2 v + n_2 (v - V) + t_2\}^2$$
$$+ \{m'_2 + \alpha'_2 u + \beta'_2 v + n_2 (v - V) + t_2\}^2 + \ldots$$
$$+ \ldots\ldots\ldots\ldots\ldots = \text{a minimum.}$$

The letters accentuated 1, 2 ... 8 appertaining to the eight Arcs.

Let U and V be the values of u and v which belong to the mean ellipse. These values, then, must be put for u and v in the two equations produced by differentiating the above with respect to u and v. We have

$$\alpha_1 (m_1 + \alpha_1 U + \beta_1 V + t_1) + \alpha'_1 (m'_1 + \alpha'_1 U + \beta'_1 V + t_1) + \ldots$$
$$+ \alpha_2 (m_2 + \alpha_2 U + \beta_2 V + t_2) + \alpha'_2 (m'_2 + \alpha'_2 U + \beta'_2 V + t_2) + \ldots$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0;$$

and

$$n_1 t_1 + (\beta_1 + n_1) (m_1 + \alpha_1 U + \beta_1 V + t_1)$$
$$+ (\beta'_1 + n_1) (m'_1 + \alpha'_1 U + \beta'_1 V + t_1) + \ldots$$

$$n_2 t_2 + (\beta_2 + n_2)(m_2 + \alpha_2 U + \beta_2 V + t_2) + (\beta'_2 + n_2)(m'_2 + \alpha'_2 U + \beta'_2 V + t_2) + \ldots$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0.$$

Let (m) be a symbol representing the sum of all the $m$es appertaining to the divisions of the same arc; and let $\Sigma(m)$ represent the sum of all these sums for all the arcs; and similarly for other quantities besides m. Then the above equations become

$$\Sigma(m\alpha) + \Sigma(\alpha^2) U + \Sigma(\alpha\beta) V + \Sigma t(\alpha) = 0,$$

and

$$\left.\begin{aligned} &\Sigma(m\beta) + \Sigma(\alpha\beta) U + \Sigma(\beta^2) V + \Sigma t(\beta) \\ &+ \Sigma n(m) + \Sigma n(\alpha) U + \Sigma n(\beta) V + \Sigma t n i \end{aligned}\right\} = 0,$$

i being the number of stations on the representative arc.

The numerical quantities involved in the first two lines of these equations have been calculated in the article on the Figure of the Earth in the British Ordnance Survey Volume; and the remaining ones have been calculated by the author, and the whole gathered together in an article published in the Proceedings of the Royal Society, Vol. XII. No. 64, p. 253. The numerical quantities are there substituted, the equations are solved, and the following results obtained.

$$a = 20928627 + 1057{\cdot}8t_1 + 342{\cdot}9t_2 + 152{\cdot}3t_3 + 27{\cdot}3t_4 + 93{\cdot}6t_5 + 8{\cdot}8t_6 + 63{\cdot}7t_7 + 62{\cdot}9t_8,$$

$$b = 20849309 - 3762{\cdot}6t_1 - 334{\cdot}3t_2 - 661{\cdot}3t_3 - 101{\cdot}5t_4 - 372{\cdot}6t_5 - 14{\cdot}0t_6 - 249{\cdot}3t_7 - 249{\cdot}1t_8.$$

$$\epsilon = \frac{1}{263{\cdot}9}\{1 + 0{\cdot}0608t_1 + 0{\cdot}0085t_2 + 0{\cdot}0103t_3 + 0{\cdot}0016t_4 + 0{\cdot}0059t_5 + 0{\cdot}0003t_6 + 0{\cdot}0039t_7 + 0{\cdot}001639t_8\}.$$

PROP. *To show the degree of uncertainty local attraction, if not allowed for, introduces into the problem of the Figure of the Earth.*

133. The formulæ deduced in the last Article for the semiaxes and ellipticity of the mean figure of the earth show

us, that the effect of local attraction upon the final numerical results may be very considerable: for example, a deflection of the plumb-line of only 5″ at the standard station (St Agnes) of the Anglo-Gallic arc would introduce a correction of about one mile to the length of the semi-major-axis, and more than three miles to the semi-minor-axis. If the deflection at the standard station (Damargida) of the Indian Great Arc be what the mountains and ocean make it (without allowing any compensating effect from variations in density in the crust below, which no doubt exist, but which are altogether unknown), viz. about 17″·3 (Art. 62, Ex. 1 and 2), the semiaxes will be subject to a correction, arising from this cause alone, of half a mile and two miles. This is sufficient to show how great a degree of uncertainty local attraction, if not allowed for, introduces into the determination of the mean figure. As long as we have no means of ascertaining the amount of local attraction at the several standard stations of the arcs employed in the calculation, this uncertainty regarding the mean figure, as determined by geodesy, must remain.

PROP. *To state the result of applying the same principles to each of the three long arcs, the Anglo-Gallic, Russian, and Indian, and to obtain a Mean Figure of the Earth from them.*

134. The first three of the eight arcs which have been used in the above calculation, viz. the Anglo-Gallic, Russian, and Indian, are of considerable length. If the method of the last Article is applied to each of these separately we shall obtain three pairs of values of the semi-axes, involving the three unknown expressions for local attraction at the three standard stations. If local attraction be neglected, these pairs will differ slightly from each other, suggesting an idea, which we shall notice in Art. 141, that the equator is not a circle. But local attraction is more important in its effects than any other known cause of derangement, and must not be neglected; and the inference that the equator is not a circle cannot be drawn without better evidence being adduced against it. All calculations which have hitherto been made of the mean figure of the earth have gone on the hypothesis that it is an oblate spheroid, and the *à priori* argument in its favour from the fluid-theory is so overwhelming that it must not be

abandoned without sufficient evidence. In what follows we shall show, that so far from there being any evidence that the equator is not a circle, the three elliptic meridians of the Anglo-Gallic, the Russian, and the Indian arcs can be made almost precisely the same by a *very moderate* allowance for local attraction.

In the previous calculation t has represented the angle which the plumb-line makes, in the plane of the meridian, with the normal to the mean ellipse of the earth. We shall now use T as the angle which the plumb-line makes, in the plane of the meridian, with the normal to the mean ellipse of the particular arc under consideration.

The calculation will be found in the Royal Society's Proceedings already referred to. The following are the results for the Three Arcs :—

$$a_1 = 20928190 + 1577{\cdot}7\,T_1, \quad b_1 = 20847200 - 5885{\cdot}9\,T_1,$$

$$\epsilon_1 = \frac{1}{258{\cdot}4}(1 + 0{\cdot}0921\,T_1).$$

$$a_2 = 20927234 + 345{\cdot}2\,T_2, \quad b_2 = 20861620 + 2832{\cdot}4\,T_2,$$

$$\epsilon = \frac{1}{318{\cdot}9}(1 - 0{\cdot}0379\,T_2).$$

$$a_3 = 20926529 + 13862{\cdot}8\,T_3, \quad b_3 = 20855535 + 5555{\cdot}6\,T_3,$$

$$\epsilon_3 = \frac{1}{294{\cdot}8}(1 + 0{\cdot}1170\,T_3).$$

If these ellipses are made equal to each other, that is,

$$a_1 = a_2 = a_3, \quad b_1 = b_2 = b_3,$$

these formulæ give four equations of condition connecting the three quantities T_1, T_2, T_3. The most likely solutions of these four equations are found, by the method of least squares, to be

$$T_1 = -1''{\cdot}37, \quad T_2 = -2''{\cdot}22, \quad T_3 = -0''{\cdot}033.$$

When these are substituted in the semiaxes, they give

$$a_1 = 20926029, \quad a_2 = 20926468, \quad a_3 = 20926072.$$

$$b_1 = 20855264, \quad b_2 = 20855332, \quad b_3 = 20855352.$$

These three results are remarkably near each other; they differ from their average, 20926189 and 20855316, in no case

by so much as 300 feet, and in most cases by much less. We may safely infer that this average ellipse is in fact the Mean Figure of the Earth. This being the case, T_1, T_2, T_3 are the same as t_1, t_2, t_3; and therefore the deflections of the plumb-line in the meridian at the standard stations of the Anglo-Gallic, Russian, and Indian arcs are 1″·37, 2″·22, 0″·033, all in the southern direction.

135. The values, then, which we would assign to the semiaxes and ellipticity of the Mean Figure of the Earth are as follows:—

$$a = 20926189,\quad b = 20855316 \text{ feet},\quad \epsilon = \frac{1}{295{\cdot}3}.$$

We believe that the axes have never before been found taking into consideration the effect of Local Attraction.

136. When these values of t, a, and b are substituted in the formulæ for the corrections of latitude they give in fact the local deflections at all the stations on the Three Arcs. They all come out remarkably small, none of them at all to be compared with the large deflections caused by the Himmalayas and the Ocean in India. Thus even at the two extreme stations of the Great Arc of India beginning with Cape Comorin they are only $-0''{\cdot}94$ and $+1''{\cdot}34$. And it is curious that out of 13 coast-stations, in 7 of them what deflection there is is *towards* the sea. See *Proc. Roy. Soc.* Vol. XII. No. 64, p. 270.

PROP. *To deduce from the previous calculation some probable conclusions regarding the Constitution of the Earth's Crust.*

137. The first thing to be observed in the results given in the last paragraph is the very small amount of the resultant deflections at the two extremities of the Indian Arc—Punnœ close to Cape Comorin, and Kaliana the nearest station to the Himmalaya Mountains; whereas the effect of the Ocean and the Mountains has been shown to be very large. This shows that the effect of variations of density in the crust must be very great, in order to bring about this near compensation. In fact the density of the crust beneath the mountains must be less than that below the plains, and still less than that

below the ocean-bed. If solidification from the fluid state commenced at the surface, the amount of contraction in the solid parts beneath the mountain-region has been less than in the parts beneath the sea. In fact, it is this unequal contraction which appears to have caused the hollows in the external surface which have become the basins into which the waters have flowed to form the ocean. As the waters flowed into the hollows thus created, the pressure on the ocean-bed would be increased, and the crust, so long as it was sufficiently thin to be influenced by hydrostatic principles of floatation, would so adjust itself that the pressure on any *couche de niveau* of the fluid should remain the same. At the time that the crust first became sufficiently thick to resist fracture under the strain produced by a change in its density—that is, when it first ceased to depend for the elevation or depression of its several parts upon the principles of floatation, the total amount of matter in any vertical prism, drawn down into the fluid below to a given distance from the earth's centre, had been the same through all the previous changes. After this, any further contraction or any expansion in the solid crust would not alter the amount of matter in the vertical prism, except where there was an ocean; in the case of greater contraction under an ocean than elsewhere, the ocean would become deeper and the amount of matter greater, and in case of a less contraction or of an expansion of the crust under an ocean, the ocean would become shallower, or the amount of matter in the vertical prism less than before. It is not likely that expansion and contraction in the solid crust would affect the arrangement of matter in any other way. That changes of level do take place, by the rising and sinking of the surface, is a well-established fact, which rather favours these theoretical considerations. But they receive, we think, great support from the other fact, that the large effect of the ocean at Punnœ and of the mountains at Kaliana almost entirely disappears from the resultant deflections brought out by the calculations.

138. This theory, that the wide ocean has been collected on parts of the earth's surface where hollows have been made by the contraction and therefore increased density of the crust below, is well illustrated by the existence of a whole hemisphere of water, of which New Zealand is the pole, in stable

equilibrium. Were the crust beneath only of the same density as that beneath the surrounding continents, the water would be drawn off by attraction and not allowed to stand in the undisturbed position it now occupies. (See Art. 155.)

139. We have, in what goes before, supposed that, in solidifying, the crust contracts and grows denser, as this appears to be most natural, though, after the solid mass is formed, it may either expand or contract, according as an accession or diminution of heat may take place. If, however, in the process of solidifying, the mass becomes lighter, the same conclusion will follow—the mountains being formed by a greater degree of expansion of the crust beneath them, and not by a less contraction, than in the other parts of the crust. It may seem at first difficult to conceive how a crust could be formed at all, if in the act of solidification it becomes heavier than the fluid on which it rests; for the equilibrium of the heavy crust floating on a lighter fluid would be unstable, and the crust would sooner or later be broken through, and would sink down into the fluid, which would overflow it. If, however, this process went on perpetually, the descending crust, which was originally formed by a loss of heat radiated from the surface into space, would reduce the heat of the fluid into which it sank, and after a time a thicker crust would be formed than before, and the difficulty of its being broken through would become greater every time a new one was formed. Perhaps the tremendous dislocation of stratified rocks in huge masses with which a traveller in the mountains, especially in the interior of the Himmalaya region, is familiar, may have been brought about in this way. The catastrophes, too, which geology seems to teach have at certain epochs destroyed whole species of living creatures, may have been thus caused, at the same time breaking up the strata in which those species had for ages before been deposited as the strata were formed. These phenomena must now long have ceased to occur, at any rate on a very extensive scale, as Mr Hopkins's and Professor W. Thomson's investigations appear to prove that the crust is very thick. See Art. 114, 5.

140. The circumstance already noticed, that at seven coast-stations out of thirteen the deflection is towards the sea, seems to bear testimony to the truth of the theory, that the crust below the ocean must have undergone greater contraction than

other parts. The deflection towards the land at the other six coast-stations can of course easily be understood without at all calling in question the theory. The proximity of the land may easily be conceived sufficient to counteract any effect of the more distant parts of the ocean. It is the fact of even *some* of the deflections being towards the sea, that bears testimony to the theory, while the others offer no argument to the contrary.

The least, then, that can be gathered from the deflections of these coast-stations is, that they present no obstacle to the theory so remarkably suggested by the facts brought to light in India, viz., that mountain-regions and oceans on a large scale have been produced by the contraction of the materials, as the surface of the earth has passed from a fluid state to a condition of solidity—the amount of contraction beneath the mountain-region having been less than that beneath the ordinary surface, and still less than that beneath the ocean-bed, by which process the hollows have been produced into which the ocean has flowed. In fact the testimony of these coast-stations is rather in favour of the theory, as they seem to indicate, by *excess* of attraction towards the sea, that the contraction of the crust beneath the ocean has gone on increasing in some instances still further since the crust became too thick to be influenced by the principles of floatation, and that an additional flow of water into the increasing hollow has increased the amount of attraction upon stations on its shores*.

141. General T. F. de Schubert has suggested in his *Essai d'une Determination de la veritable Figure de la Terre*, that the figure is better represented by an ellipsoid than by a spheroid. His process is this. He finds the nearest ellipses which represent the meridians of the Russian, Indian, and French arcs, the three longest which have been measured. This he does by dividing each arc into two parts and comparing the two parts with each other or with the whole. The Russian arc, divided into two at Dorpat, latitude $58^{\circ}\,23'$, gives for the minor-semiaxis 3261429 toises; the Indian arc, divided at Damargida, latitude $18^{\circ}\,3'$, gives 3261547; the French arc,

* The first part of this theory apparently confirms Mr Airy's hypothesis (*Phil. Trans.* 1855, p. 101). But his reasoning is based on the crust being thin—so thin as to be influenced in its position by the fluid below; which cannot be admitted.

divided at Carcassonne, latitude 43° 13′, gives 3260365. The first two agree very nearly. He rejects, therefore, the third, and uses the mean of the other two, giving twice the weight to the Russian that he does to the Indian: this produces 3261468 toises for the minor-semiaxis common to all meridians. With this minor-semiaxis he calculates the major-semiaxis in the Peruvian, Russian, and Indian meridians, selecting these arcs partly because of their difference of longitude. He finds the resulting semiaxes to be different, and concludes that the equator is not circular. He assumes it to be an ellipse: and finds that an ellipse with semiaxes 32726711 and 32723031 in longitude (measured from a meridian 20° west of Paris) 58° 44′ and 148° 44′ respectively will pass through those meridians at their middle points. This makes the ellipticity of the two principal meridians of the ellipsoid to be $\frac{1}{292}$ and $\frac{1}{302}$. He next computes the radii of this equatorial ellipse which correspond to the meridians of the different arcs measured in various parts of the world: these are in fact the semi-major-axes of the meridians of those arcs. With the semiaxes of the several meridians thus determined he computes the geodetic amplitudes of the several arcs and compares them with the astronomical: the following is the result:

G. amp. − *A. amp.*		*G. amp.* − *A. amp.*	
Peruvian arc	0″·077	Cape of Good Hope	−0″·442
Pennsylvanian	−6 ·687	Prussian	1 ·267
English (entire arc)	0 ·736	Russian	−1 ·289
French	−1 ·607	Indian	1 ·619

The Pennsylvanian, as is well known, deserves no *à priori* confidence. The other quantities are small. The Indian arc shows a difference, however, of 3″·77 (see Art. 120), more than double the difference here given. Also the measure of the French arc has been rejected without any apparent reason. So that the approximate appearance of the result must be regarded rather as accidental. Mr Airy (from whose notice of the work in the Monthly Notices of the Astronomical Society, Vol. XX., the above remarks have been gathered) recommends that the polar semiaxis should be determined, with the other semiaxes, by a combination of the lengths of all the arcs, introducing also the latitudes of middle stations.

142. A similar calculation was afterwards made by Capt. Clarke, with Bessel's method (see *Memoirs Roy. Ast. Soc.* 1859—60, p. 25); but he neglects local attraction, as General de Schubert has done, although it is a disturbing cause of much more importance than any which the method of least squares is used to eliminate. In a subsequent paper indeed the General points out that local attraction may greatly modify, if not altogether destroy, the discrepancies between the different meridians (see *Monthly Notices of Roy. Ast. Soc.* No. 6, April 13, 1860, p. 264), a result which our calculation based upon the modification of Bessel's method fully confirms.

The following calculation shows the same in a simpler way.

PROP. *In comparing two divisions of an arc of meridian, to find the effect of a small deflection of the plumb-line at the middle station on the resulting axes.*

143. Let $\lambda + \lambda'$ be the astronomical amplitude of the whole arc, and λ and λ' the amplitudes of its two divisions. Then for determining the form of the meridian, we have, by Art. 124,

$$\frac{a-b}{2} = \frac{1}{3}\frac{\frac{s}{\lambda} - \frac{s'}{\lambda'}}{\cos 2m' - \cos 2m}, \qquad \frac{a+b}{2} = \frac{\frac{s}{\lambda}\cos 2m' - \frac{s'}{\lambda'}\cos 2m}{\cos 2m' - \cos 2m}.$$

Suppose the latitude of the middle station is wrong, owing to unknown local attraction, by the quantity $\delta\lambda$: then as $\lambda + \lambda'$ is, by hypothesis, correctly determined, $\delta\lambda' = -\delta\lambda$;

$$\therefore \frac{1}{2}(\delta a - \delta b) = -\frac{1}{3}\frac{\frac{s}{\lambda^2} + \frac{s'}{\lambda'^2}}{\cos 2m' - \cos 2m}\delta\lambda,$$

$$\frac{s}{\lambda} = \frac{s'}{\lambda'} = \frac{a+b}{2},$$

(by Art. 124), neglecting the ellipticity;

$$\therefore \frac{\delta a - \delta b}{a+b} = -\frac{1}{3}\frac{1}{\cos 2m' - \cos 2m}\frac{\lambda+\lambda'}{\lambda'}\frac{\delta\lambda}{\lambda}.$$

By a similar process we get

$$\frac{\delta a + \delta b}{a+b} = -\frac{\cos 2m' + \frac{\lambda}{\lambda'}\cos 2m}{\cos 2m' - \cos 2m}\frac{\delta\lambda}{\lambda}.$$

Ex. 1. In the Russian arc $\lambda = 13^\circ 1' = 46860''$ from Staro-Nekrassowka to Dorpat, $\lambda' = 12^\circ 17' = 44220''$ from Dorpat to Fuglenæs; and twice the middle latitudes are $103^\circ 43'$ and $129^\circ 3'$.

If $\delta\lambda = 1''$ only,

$$\frac{\delta a - \delta b}{a+b} = 0{\cdot}0000373, \quad \frac{\delta a + \delta b}{a+b} = -0{\cdot}0000265;$$

$$\therefore\ \delta a = \frac{a+b}{2} \times 0{\cdot}0000108 = 20890790 \times 0{\cdot}0000108 = 226 \text{ feet},$$

$$\delta b = -\frac{a+b}{2} \times 0{\cdot}0000638 = -1333 \text{ feet}.$$

Ex. 2. In the Indian arc, divided at Damargida,

$$\lambda = 11^\circ 27' 33'' = 41253''$$

from Kaliana to Damargida,

$$\lambda' = 9^\circ 53' 44'' = 35624''$$

from Damargida to Punnœ:

also $2m = 47^\circ 34'$, $2m' = 26^\circ 13'$. If $\delta\lambda = 1''$,

$$\frac{\delta a - \delta b}{a+b} = -0{\cdot}0000788, \quad \frac{\delta a + \delta b}{a+b} = -0{\cdot}0001838,$$

$$\delta a = -\frac{a+b}{2} \times 0{\cdot}0002626 = -5486 \text{ feet},$$

$$\delta b = -\frac{a+b}{2} \times 0{\cdot}0001050 = -219 \text{ feet}.$$

These are large quantities; and if they are so large for only 1″ of local attraction, they may be in fact much larger than this, without our having any means of knowing it. We have already shown (Art. 62) that there may be much larger deflections than 1″ without any visible cause to produce them. The calculations referred to in the last Article regarding the elliptical form of the equator are, therefore, not to be considered as trustworthy.

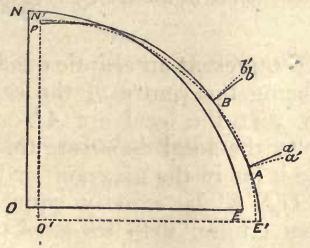
§ 2. *The form of separate parts of the surface.*

144. What has gone before leads to the determination of only the Mean Figure of the Earth. Our knowledge, however, of the surface—diversified as it is with mountains, plains, and oceans—is sufficient to show that particular parts of the surface depart from this mean figure.

We have shown already that the large effects of the Himmalayas and the Ocean in India are very nearly compensated for by variations in density in the crust. The residual deflections, however, are not to be overlooked. It is to the consideration of these that we now call the attention of the student. In the course of our remarks some things will be explained which probably have not been so thoroughly understood in what has gone before as they will be now.

PROP. *To explain what is meant by the Sea-level, and to point out its use.*

145. In the diagram suppose A is the station from which we commence: and suppose the dark line AB to be the curve



in which still water would lie, if a canal were cut from the sea along the meridian through A northwards, and the sea were allowed to flow into it. This curve is called the SEA-LEVEL. Where the level changes owing to the ebb and flow of the tide the mean is taken.

The plumb-line at every place along this curve hangs at right angles to the curve at that place; because it is one condition of fluid equilibrium, that the resultant force at any point of the fluid surface acts in the normal at that point (Art. 75). This level-curve will partake, therefore, of all the irregularities of the plumb-line caused by local attraction. It indicates the general form of the surface, altered as it has become, since the earth ceased to be a fluid mass, by the upheavings and sinkings which geology teaches us have most certainly taken place.

It is this curve which is meant when we speak of the Arc of Meridian, and it is the work of the Trigonometrical Survey to determine its form, and to measure the elevations and depressions of places on the meridian with reference to it. A and B are in fact points in which verticals through these stations cut this level-curve, and are not necessarily the places themselves, which may be some feet above or below them. The exact contour of the earth's visible surface is obtained by finding the form of the level-curve or arc of meridian, and also the elevations or depressions of places, above or below this curve. The level-curve is not necessarily an ellipse: indeed most likely it is not: but as it evidently does not differ much from a circle short portions of it may be represented very well by an elliptic arc of small ellipticity.

PROP. *To explain what is meant by Astronomical and Mean Amplitudes.*

146. Let ONE represent an elliptic quadrant of the earth's mean figure, O being the centre of the earth. It does not necessarily follow that the local arc AB should lie on this quadrant; owing to the local departure from the mean figure AB may lie above it (as in the diagram) or below it. Let the dotted quadrant $O'N'E'$ be exactly equal to the quadrant ONE of the mean ellipse, with its axes parallel to those of

that ellipse, and its centre O' so situated that the circumference of the ellipse passes through both A and B. If the curvature of the arc AB is not that of the mean arc, but if some other line represent it, as the continuous line AB, then the plumb-line will hang in the normals aA, bB to this line, and the angle they include is the observed or *astronomical amplitude* of the arc, because it is measured by the corresponding arc in the heavens, defined by the points in which the plumb-line at its extremities intersects the celestial vault.

The dotted line AB represents the mean arc, and the inclination of the dotted normals $a'A$, $b'B$ measures the *mean amplitude* of the arc, and can be calculated by the formula of Art. 123, when we know the length of the mean arc, as well as of the mean axes.

The astronomical amplitude can, therefore, always be obtained by observation. The mean amplitude could not be thus obtained, unless we happened to know that the actual arc coincided with the mean. We proceed to show how in all cases the mean amplitude may be determined by means of the geodetic arc measured by the Survey and the mean axes found as already described.

PROP. *To prove that the lengths of the mean arc and of the geodetic arc of meridian between two places, as much as twelve degrees and a half apart, differ by an insensible quantity: and to show how the mean amplitude can be obtained by this theorem.*

147. Let s be the length of the elliptic arc between the stations, l and l' the observed latitudes of the extremities, λ and m the amplitude and middle latitude. Let c be the chord, r and θ, r' and θ' the polar co-ordinates from the centre of the ellipse to the extremities of the arc, a and b the semiaxes;

$$\therefore\ c^2 = r^2 + r'^2 - 2rr'\cos(\theta - \theta') = 2rr'\{1 - \cos(\theta - \theta')\} + (r - r')^2,$$

$$r = a(1 - \epsilon\sin^2 l),\ r' = a(1 - \epsilon\sin^2 l').$$

Also $\tan\theta = (1 - 2\epsilon)\tan l,\ \theta = l - \epsilon\sin 2l;$

$$\therefore\ \theta - \theta' = \lambda - 2\epsilon\sin\lambda\cos 2m;$$

$$\therefore\ 1-\cos(\theta-\theta')=1-\cos\lambda-2\epsilon\sin^2\lambda\cos 2m$$

$$=2\sin^2\frac{1}{2}\lambda\{1-2\epsilon(1+\cos\lambda)\cos 2m\};$$

$$\therefore\ c^2=4a^2\sin^2\frac{1}{2}\lambda\{1-2\epsilon(1+\cos\lambda)\cos 2m-\epsilon(\sin^2 l+\sin^2 l')\}$$

$$=4a^2\sin^2\frac{1}{2}\lambda[1-\epsilon\{1+(2\cos\lambda)\cos 2m\}];$$

$$\therefore\ \sin\frac{1}{2}\lambda=\frac{c}{2a}\left[1+\frac{1}{2}\epsilon\{1+(2+\cos\lambda)\cos 2m\}\right];$$

$$\therefore\ \frac{\lambda}{2}=\sin^{-1}\frac{c}{2a}+\frac{1}{2}\epsilon\{1+(2+\cos\lambda)\cos 2m\}\frac{c}{\sqrt{4a^2-c^2}}$$

$$=\sin^{-1}\frac{c}{2a}+\frac{1}{2}\epsilon\{1+(2+\cos\lambda)\cos 2m\}\tan\frac{1}{2}\lambda.$$

Now $s=a\left(1-\frac{1}{2}\epsilon\right)\lambda-\frac{3}{2}a\epsilon\sin\lambda\cos 2m$, by Art. 122,

$$\therefore\ s=a(2-\epsilon)\sin^{-1}\frac{c}{2a}+a\epsilon\{1+(2+\cos\lambda)\cos 2m\}\tan\frac{1}{2}\lambda$$

$$-\frac{3}{2}a\epsilon\sin\lambda\cos 2m$$

$$=(a+b)\sin^{-1}\frac{c}{2a}+(a-b)\left\{1+\frac{1}{2}(1-\cos\lambda)\cos 2m\right\}\tan\frac{1}{2}\lambda.$$

Taking the variations of s with respect to a and b, c being constant, as also λ and m because they occur in small terms, we have the difference in length of two arcs joining the stations and belonging to different ellipses, only having their axes parallel.

$$\therefore\ \delta s=(\delta a+\delta b)\sin^{-1}\frac{c}{2a}-\frac{a+b}{a}\frac{c\delta a}{\sqrt{4a^2-c^2}}$$

$$+(\delta a-\delta b)\left\{1+\frac{1}{2}(1-\cos\lambda)\cos 2m\right\}\tan\frac{1}{2}\lambda.$$

The terms being small we may approximate;

$$\therefore\ \delta s = (\delta a + \delta b)\frac{1}{2}\lambda - 2\tan\frac{1}{2}\lambda\,.\,\delta a$$

$$+ (\delta a - \delta b)\left\{1 + \frac{1}{2}(1-\cos\lambda)\cos 2m\right\}\tan\frac{1}{2}\lambda$$

$$= (\delta a+\delta b)\left(\frac{1}{2}\lambda - \tan\frac{1}{2}\lambda\right) + (\delta a - \delta b)\frac{1}{2}\tan\frac{1}{2}\lambda\,(1-\cos\lambda)\cos 2m$$

$$= (\delta a + \delta b)\,P + (\delta a - \delta b)\,Q\cos 2m, \text{ suppose,}$$

$$= (P + Q\cos 2m)\,\delta a + (P - Q\cos 2m)\,\delta b;$$

δa and δb are two arbitrary increments of a and b. We will find the least values of these which will produce a given increase δs to the arc: that is, the values which make $\delta a^2 + \delta b^2$ a minimum.

$$\therefore\ \delta a^2 + \left\{\frac{\delta s - (P + Q\cos 2m)\,\delta a}{P - Q\cos 2m}\right\}^2 = \text{a minimum};$$

$$\therefore\ \{(P - Q\cos 2m)^2 + (P + Q\cos 2m)^2\}\,\delta a$$
$$= (P + Q\cos 2m)\,\delta s;$$

$$\therefore\ \delta a = \frac{P + Q\cos 2m}{P^2 + Q^2\cos^2 2m}\,\frac{\delta s}{2},$$

$$\delta b = \frac{P - Q\cos 2m}{P^2 + Q^2\cos^2 2m}\,\frac{\delta s}{2},$$

$$\delta a^2 + \delta b^2 = \frac{1}{P^2 + Q^2\cos^2 2m}\,\frac{\delta s^2}{2}.$$

This is least when $m = 0$ or 90°; then

$$\delta a = \frac{P \pm Q}{P^2 + Q^2}\,\frac{\delta s}{2},\quad \delta b = \frac{P \mp Q}{P^2 + Q^2}\,\frac{\delta s}{2};$$

$$\text{and}\quad \delta a \sim \delta b = \frac{Q\delta s}{P^2 + Q^2}.$$

Let one of the ellipses be equal to the ellipse of the earth's mean figure, a and b being the semi-axes; then δa and δb will be the excess (or defect, if negative) of the semi-axes of the other ellipse: this latter ellipse is the ellipse which most nearly coincides with the actual arc s of the level curve and therefore represents it. The first ellipse is not necessarily the mean ellipse itself, but is only equal to it in dimensions and parallel to it in position; for the actual arc may lie above or below the mean ellipse. The result of this is, that the arc of the mean ellipse which corresponds with s of the actual measured arc will not necessarily have precisely the same middle latitude, although the chord c is of the same length. But as the middle latitude will differ only by a quantity of the order of the ellipticity this difference will not appear in the result because we neglect the square of the ellipticity.

We will put $\delta s = \text{arc } 1'' = 0{\cdot}0193$ mile, 1^0 being $69{\cdot}5$ miles: and will find the value of λ which will make $\delta a \sim \delta b$ as large as the whole compression of the earth's pole, viz. 13 miles. This gives

$$P^2 + Q^2 \div Q = 0{\cdot}0193 \div 13 = 0{\cdot}0015,$$

$$\text{or } \left(\frac{\lambda}{2} - \tan\frac{\lambda}{2}\right)^2 + \frac{1}{4}\tan^2\frac{\lambda}{2}(1-\cos\lambda)^2$$

$$= 0{\cdot}00075 \tan\frac{\lambda}{2}(1-\cos\lambda).$$

A slight inspection of this equation shows that λ must be small. Expand in powers of λ; then

$$\left(\frac{1}{9}+1\right)\left(\frac{\lambda}{2}\right)^3 = 0{\cdot}0015, \quad \text{or } \left(\frac{\lambda}{2}\right)^3 = 0{\cdot}00135;$$

$$\therefore \lambda = 0{\cdot}22 \text{ (in arc)} = 0{\cdot}22 \times 57^0{\cdot}3 \text{ (in degrees)} = 12^0{\cdot}6.$$

This shows, that in an arc of meridian as much as twelve degrees and a half in length it would require a departure from the mean ellipse equal to the whole actual compression of the pole of the earth in order to produce so slight a difference in the length as 1″. Hence we may conclude that the difference in length between the mean arc and the actual

arc is in fact an insensible quantity, since an extravagant hypothesis regarding the departure of the form of the arc in question from the mean form will not produce a difference of length of more than 1″.

148. The property here proved shows us at once how the mean amplitude of the arc may be found. By the formula in Art. 122 the mean amplitude may be calculated from the mean axes and the length of the mean arc when it is found. But the property now proved shows that this length is sensibly the same as the length of the geodetic arc, that is, the arc actually measured in the Survey, even though it may be altered in position by geological changes. This latter, then, may be used in the formula instead of the length of the mean arc, which but for this property would be unknown.

149. From what goes before it appears, that the difference between the astronomical amplitude and the mean amplitude thus found measures exactly the difference of meridian deflection caused by local attraction at the two extremities of the arc. The following PROP. will illustrate this.

PROP. *To estimate the relative amount of local attraction in the plane of the meridian at stations on the Indian Arc.*

150. The stations we shall take are Kaliana (29° 30′ 48″), Kalianpur (24° 7′ 11″), Damargida (18° 3′ 15″), and Punnœ (8° 9′ 31″). The lengths of the arcs connecting these stations (see Volume of the British Ordnance Survey, p. 757, where the data are all brought together) are 1961138, 2202905, and 3591784 feet respectively. By Art. 123, Cor. 1, we have the following formula:

$$\frac{s}{\lambda} = \frac{a+b}{2} - 3\,\frac{a-b}{2}\cos 2m,$$

or, since we neglect the square of the ellipticity,

$$\lambda = \frac{2s}{a+b}\left(1 + \frac{3}{2}\epsilon\cos 2m\right),$$

a, b, and ϵ are 20926180, 20855316, $\frac{1}{295{\cdot}3}$ (Art. 135).

		Kaliana to Kalianpur.	Kalianpur to Damargida.	Damargida to Punnœ.
$\log \frac{3}{2}$	=	0·1760913	0·1760913	0·1760913
$\log \epsilon$	=	$\bar{3}$·5297366	$\bar{3}$·5297366	$\bar{3}$·5297366
$\log \cos 2m$	=	$\bar{1}$·7730214	$\bar{1}$·8698830	$\bar{1}$·9528700
		$\bar{3}$·4788493	$\bar{3}$·5757109	$\bar{3}$·6586979
$1+\frac{3}{2}\epsilon\cos 2m$	=	1·0030120	1·0037645	1·0045572
$\log(1+\frac{3}{2}\epsilon\cos 2m)$	=	0·0013061	0·0016318	0·0019746
$\log 2s$	=	6·5935392	6·6440258	6·8563402
$\log(a+b)$	=	7·6209840	7·6209840	7·6209840
$\therefore \log \lambda$	=	$\bar{2}$·9738613	$\bar{1}$·0246736	$\bar{1}$·2373308
$\log \sin 1''$	=	$\bar{6}$·6855749	$\bar{6}$·6855749	$\bar{6}$·6855749
$\log \lambda''$	=	4·2882864	4·3390987	4·5517557
λ''	=	19421·66	21832·36	35625·66
		$=5^0\ 23'\ 41''{\cdot}66$	$6^0\ 3'\ 52''{\cdot}36$	$9^0\ 53'\ 45''{\cdot}66$

The results may be thus tabulated:

Arcs.	Astronomical Amplitudes.	Mean Amplitudes.	Differences.
	° ′ ″	° ′ ″	″
Kaliana to Kalianpur	5 23 37·06	− 5 23 41·26	= −4·20*
Kalianpur to Damargida	6 3 55·97	− 6 3 52·36	= +3·61
Damargida to Punnœ.....	9 53 44·16	− 9 53 45·66	= −1·50

Each of these differences measures the difference of local attraction in the meridian at the two stations at the extremities of the arc. They do not lead to the absolute amount of local attraction, but only to the difference of its amount in passing from one station to the next (see Art. 149).

151. The quantities above deduced are independent of any theory regarding the structure of the earth's mass. We may, however, endeavour to trace these resulting effects to their causes. In a former part of this treatise (Art. 62) it has been explained that two visible causes exist producing deflection, viz. the mountain mass on the north of India and the vast ocean on the south. It has also been shown (Art. 64) that a hidden cause of deflection may lie below, in the variation of the density of the earth's crust. The effect of the two visible causes has been estimated approximately by the author as follows (*Phil. Trans.* 1859):—

* Colonel Everest brings out the first and second of these quantities $-5''{\cdot}24$ and $+3''{\cdot}79$; but he works with different mean axes. We have used those of Art. 135.

Deflections northwards at Kaliana,	Kalianpur,	Damargida,	Punnœ,
	"	"	"
Caused by the Mountains... 27·98	12·05	6·79	2·50 *?
Caused by the Ocean ..... 6·18	9·00	10·44	19·71
Totals 34·16	21·05	17·23	22·21

By these quantities the latitudes are diminished. Therefore the errors in the amplitudes are

$$-13''{\cdot}11,\ -3''{\cdot}82,\ +4''{\cdot}98.$$

These differ considerably from the differences of amplitude deduced from the arcs in the last Article. This shows us that there must be irregularities in the density of the crust below: their effect on the amplitudes is shown as follows:—

Differences of amplitude determined in last Article	$-4''{\cdot}20,\ +3''{\cdot}61,\ -1''{\cdot}50.$
Effect of mountains and ocean	$-13{\cdot}11,\ -3{\cdot}82,\ +4{\cdot}98.$
Consequent effect of the hidden causes in the crust below	$+8{\cdot}91,\ +7{\cdot}43,\ -6{\cdot}48.$

The hidden cause increases the amplitudes of the northern and middle of the three divisions of the Great Indian Arc, that is, makes the plumb-lines hang at a greater angle to each other; and diminishes the amplitude of the southern division, or makes the plumb-lines at its extremities hang less inclined to each other. An infinite variety of hypothetical arrangements of the materials of the crust may be conceived so as to produce this effect. The general result pointed to by this calculation is quite in accordance with the speculations of Art. 137; as the diminished attraction of the less dense parts of the crust below the mountains would, as it were, let the plumb-line go at the northern end of the arc, and therefore increase its inclination to the plumb-line in the middle parts; and the increased density below the ocean would produce the opposite effect in the southern portion of the great arc.

PROP. *To prove that the length of a mean arc of longitude is sensibly the same as the geodetically measured arc, if it do not exceed fifteen degrees in length.*

* This amount was not calculated in the Paper in the *Philosophical Transactions* alluded to above, as it was not there required. It has been since roughly obtained, in the same manner as the others, for the present purpose.

152. Let S be the length of the arc, l the latitude, L the longitudinal amplitude (i.e. the difference of the longitudes of its two extremities), c the chord. Then by Art. 123, Cor. 3,

$$S = L \cos l \{a + (a - b) \sin^2 l\},$$

$$c = 2 \cos l \{a + (a - b) \sin^2 l\} \sin \frac{1}{2} L.$$

When a and b vary, c and l remain constant, but S and L vary. Hence

$$\delta S = \delta L \cos l \{a + (a - b) \sin^2 l\} + L \cos l \{\delta a + (\delta a - \delta b) \sin^2 l\},$$

$$0 = \{a + (a - b) \sin^2 l\} \cos \frac{1}{2} L \, \delta L$$

$$+ 2 \{\delta a + (\delta a - \delta b) \sin^2 l\} \sin \frac{1}{2} L;$$

$$\therefore \delta S = \left(L - 2 \tan \frac{1}{2} L\right) \cos l \{\delta a + (\delta a - \delta b) \sin^2 l\};$$

$$\therefore \delta a + (\delta a - \delta b) \sin^2 l = \frac{\delta S}{\left(L - 2 \tan \frac{1}{2} L\right) \cos l} = n, \text{ suppose:}$$

δa and δb are arbitrary increments of a and b and produce the increment δS in the arc of longitude. We will find the least values of δa and δb, or those which make $\delta a^2 + \delta b^2$ a minimum;

$$\therefore \sin^4 l \, \delta a^2 + \{(1 + \sin^2 l) \, \delta a - n\}^2 = \text{a minimum};$$

$$\therefore \{\sin^4 l + (1 + \sin^2 l)^2\} \, \delta a = n \, (1 + \sin^2 l);$$

$$\therefore \delta a = \frac{(1 + \sin^2 l) \, n}{\sin^4 l + (1 + \sin^2 l)^2}, \quad \delta b = -\frac{\sin^2 l \, . \, n}{\sin^4 l + (1 + \sin^2 l)^2};$$

$$\therefore \delta a^2 + \delta b^2 = \frac{n^2}{\sin^4 l + (1 + \sin^2 l)^2}$$

$$= \frac{\delta S^2}{\cos^2 l \{\sin^4 l + (1 + \sin^2 l)^2\} \left\{L - 2 \tan \frac{1}{2} L\right\}^2}.$$

This is least when $l = 0$; then

$$\delta a = n,\ \delta b = 0,\ \delta a - \delta b = n = \frac{\delta S}{L - 2\tan\frac{1}{2}L}.$$

Now put $\delta a \sim \delta b = 13$ miles, $\delta S =$ arc $1''$ of a great circle $= 0{\cdot}0193$ mile;

$$\therefore\ L - 2\tan\frac{1}{2}L = 0{\cdot}0193 \div 13 = 0{\cdot}0015.$$

This shows that L must be small. Expanding we have

$$L^3 = 0{\cdot}018,\ L = 0{\cdot}262 \text{ (in arc)} = 0{\cdot}262 \times 57^{\circ}{\cdot}3 = 15^{\circ},$$

This shows that in an arc of longitude as much as fifteen degrees long (the length in miles depending, of course, on the latitude) it would require a departure from the mean ellipse equal to the whole actual compression of the pole of the earth to produce a difference in the length of the arc of only 0·0193 mile, or 102 feet. If it require so extravagant an hypothesis regarding the departure of the form of the arc from the mean form to produce so small a difference in the length, we may conclude that the actual difference in length of the actual arc and the mean arc of longitude is insensible, if the arc be no longer than fifteen degrees.

PROP. *To explain what effect local attraction will have upon the mapping of a country.*

153. If the distances of places on the earth's surface referred to the mean spheroid were accurately known in miles, then by the use of the formulæ in Art. 123, and the mean axes the differences of latitude and longitude might be accurately determined, and the places laid down accordingly in a map would have their relative positions correctly assigned. But we have no direct means of ascertaining these distances. In the Propositions of Arts. 147, 152, however, it has been shown that the actual lengths of arcs measured by the Survey (that is, on the disturbed spheroid, so to speak,) differ from the lengths of the arcs on the mean spheroid by inappreciable quantities, if the

arcs are not chosen inordinately long, a thing which is never done. These measured arcs may therefore be used in this calculation instead of the mean arcs; and this convenient result is arrived at, that the relative position of places laid down on a map as determined by the Survey operations is not sensibly affected by any deviations of the form of the surface from the mean form, caused by those upheavings and depressions which geology shows us have undoubtedly taken place. The position of the map itself on the mean terrestrial spheroid would be fixed by ascertaining the absolute latitude and longitude of some one place in it. These would, of course, be affected by local attraction.

It thus appears that a map constructed wholly from geodetical measurements will be accurate in itself, that is, the relative position of places marked down in it will be correct. But the map itself will be as much out of its place on the terrestrial spheroid as the latitude and longitude of the station which fixes the map are erroneous in consequence of local attraction at that place. Also if any place is afterwards inserted in the map by observations made upon the heavens, the place will be out of its proper position by the difference in deflection of the plumb-line at that place and at the place the latitude and longitude of which fix the map.

PROP. *To estimate the degree of departure of an arc of meridian between two stations from the curvature of the mean arc.*

154. Suppose an ellipse to be drawn through the extremities of the arc and so nearly coinciding with the arc so as to represent it. Let the origin of co-ordinates be very near the centre of this ellipse; r and θ, r' and θ' polar co-ordinates to the extremities of the arc from the centre of the ellipse; α and β rectangular co-ordinates to the centre of the ellipse, and therefore very small quantities. Hence the equation to this ellipse is

$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1,$$

$$\therefore\ x^2+y^2 \text{ or } r^2=a^2+2\alpha x+2\beta y-2\epsilon\,(a^2-x^2)$$

$$=a^2+2a\alpha\cos\theta+2a\beta\sin\theta-2a^2\epsilon\sin^2\theta;$$

$$\therefore\ r = a + \alpha\cos\theta + \beta\sin\theta - ae\sin^2\theta.$$

Let R, C, C' be the values of r at the mid-latitude and at the extremities of the arc;

$$\therefore\ R = a + \alpha\cos m + \beta\sin m - (a-b)\sin^2 m,$$
$$C = a + \alpha\cos l + \beta\sin l - (a-b)\sin^2 l,$$
$$C' = a + \alpha\cos l' + \beta\sin l' - (a-b)\sin^2 l'.$$

Multiply by 1, M, and N; add, and make the coefficients of α and β vanish;

$$\therefore\ \cos m + M\cos l + N\cos l' = 0,\ \sin m + M\sin l + N\sin l' = 0;$$

$$\therefore\ M = -\frac{\sin(m-l)}{\sin(l'-l)} = -\frac{1}{2}\sec\frac{1}{2}\lambda = N;$$

$$R + MC + NC'$$
$$= a(1 + M + N) - (a-b)(\sin^2 m + M\sin^2 l + N\sin^2 l')$$
$$= a(1+2M) - \frac{1}{2}(a-b)\{1 - \cos 2m + 2M(1 - \cos\lambda\cos 2m)\}$$
$$= \frac{1}{2}(a+b)(1+2M) + \frac{1}{2}(a-b)(1 + 2M\cos\lambda)\cos 2m$$
$$= \frac{1}{2}(a+b)\left(1 - \sec\frac{1}{2}\lambda\right) + \frac{1}{2}(a-b)\left(1 - \sec\frac{1}{2}\lambda\cos\lambda\right)\cos 2m.$$

Let δa and δb be the excess of the semi-axes of the actual arc above the axes of an ellipse equal to the mean ellipse and passing through the extremities of the arc, the axes of the two ellipses being parallel. Then taking the variations, the distance required, or δR,

$$= \frac{\delta a + \delta b}{2}\left(1 - \sec\frac{\lambda}{2}\right) + \frac{\delta a - \delta b}{2}\left(1 - \sec\frac{\lambda}{2}\cos\lambda\right)\cos 2m$$
$$= -\frac{\lambda^2}{16}(\delta a + \delta b) + \frac{3\lambda^2}{16}(\delta a - \delta b)\cos 2m, \text{ neglecting } \lambda^4 \ldots$$

155. Ex. Let the arc be that between Kaliana (29° 30′ 48″) and Damargida (18° 3′ 13″): and let it be supposed to be part of the ellipse deduced in Art. 124, Ex. 2.

In this case $\delta a = 56959$, $\delta b = 19695$, (see Art. 135);

$\therefore\ \lambda = 11^{0}\,27'\,33'' = 0{\cdot}2$ in arc, $\cos 2m = \cos 47^{0}\,34' = 0{\cdot}6747$;

$$\therefore\ \delta R = -\,0{\cdot}0025\ (\delta a + \delta b) + 0{\cdot}0050\ (\delta a - \delta b)$$
$$= 0{\cdot}0025\ \delta a - 0{\cdot}0075\ \delta b = 5 \text{ feet.}$$

Although the ellipse compared with the mean ellipse differs much in the length of its axes, yet its depression at the middle point of an arc eleven degrees long, is only 5 feet.

PROP. *Geodesy furnishes no evidence, in proof or disproof, of the upheaval or depression of the Earth's surface as suggested by geological phenomena.*

156. It might at first seem from the last Article that geodesy proves, that the position of the arc has not been sensibly changed, and that geological processes have not affected it. But it must be observed, that the comparison of the arc has been made not with the mean ellipse itself, but with an ellipse equal in dimensions to the mean ellipse and with axes parallel (because the latitudes are measured in all the ellipses from the same or parallel lines). This ellipse was so drawn as to pass through the extremities of the arc; but we have no means of knowing that the mean ellipse itself passes through those two points: it may lie above them or below them. We have no means of ascertaining the precise position of the centre of the mean ellipse. The only way of doing this is to make a geodetic measurement of the whole of one meridian from pole to pole. Till this is done we have no evidence of any particular arc lying above or below the mean, i.e. of its having been elevated or depressed. The greatest geological changes of level, therefore, are perfectly consistent with all we know by geodesy of the surface of the Earth.

157. It has been explained, that in consequence of the inequalities of the Earth's surface the observations, whether made on the pendulum or in geodetic operations, are all referred to the SEA-LEVEL; that is, to that surface which the sea would form if allowed to percolate by canals through

the continents. The sea is thus taken as the basis of our measurements; and is generally assumed to have a spheroidal form. But it is possible that these local disturbing forces, arising from attraction, may have the effect of crowding up the waters in the direction in which the forces act, so as sensibly to alter the sea-level from the spheroidal form. This we shall proceed to examine.

PROP. *To find the effect of a small horizontal disturbing force in changing the Level of the Sea.*

158. Let U be the disturbing force and du an element of the line u along which it acts. Then $\int Udu$ must be added to the potential in the equation of fluid equilibrium of Art. 75.

$$\therefore \int \frac{dp}{\rho} = V + \frac{w^2}{2} r^2 (1-\mu^2) + \int Udu = \text{const. at the surface.}$$

Putting $w^2 = m \cdot \frac{E}{a^3}$ and substituting for V from Art. 85,

$$\text{constant} = \frac{E}{r} + \left(\epsilon - \frac{m}{2}\right) \frac{Ea^2}{r^3} \left(\frac{1}{3} - \mu^2\right) + \frac{m}{2} \frac{E}{a^3} (1-\mu^2) + \int Udu.$$

When the small force U is neglected, $a \div r = 1 + \epsilon . \mu^2$, by the equation to the ellipse. Hence, neglecting small quantities of the second order, dividing by E, multiplying by a, and transposing, the above equation must become

$$\frac{a}{r} = 1 + \epsilon . \mu^2 - \frac{a}{E} \int Udu.$$

Now $\frac{1}{r} \frac{dr}{d\theta}$ is the tangent of the angle between r and the normal, $= \tan \psi$ suppose: and the angle through which the normal is thrown back by the force U

$$= \delta\psi = \delta . \tan \psi = -\delta . r \frac{d}{d\theta} \frac{1}{r} = \frac{a}{E} U \frac{du}{d\theta}.$$

Hence the element ds of the undisturbed meridian line on

the surface of the sea is elevated, on the side towards which U acts, by the space

$$ds.\,d\psi = \frac{a}{E}\,U\frac{du}{d\theta}\,ds = \frac{a^2}{E}\,Udu = \frac{U}{g}\,du;$$

$$\therefore \text{ whole elevation of the sea-level} = \frac{1}{g}\int Udu,$$

integrated between the limits.

It has been shown (Art. 136) that although there are causes (such as the Himmalayas and the Ocean) which produce a considerable amount of local attraction, yet that on the whole they very nearly balance each other. The following three examples are therefore solely for applying the formula.

159. Ex. 1. The Himmalayas attract places along the coast of Hindostan with a force varying nearly inversely as the distance from a line running E.S.E. and W.N.W. through a point in latitude 33° and longitude 77° 42′, and equal to g tan 7″ at 1020 miles distance: (see *Phil. Trans.* 1855, p. 91, 94; also 1859, p. 793). Find the effect this would have upon the sea-level between Cape Comorin and Karachi, which are about 1600 and 775 miles from this line, if there were no counteracting cause, as it is believed there is (Art. 136).

In this case $U = -g \tan 7''(1020 \div u)u$ is the distance from the line. We may take the arc for the chord. Therefore rise of sea-level from this cause

$$= 1020 \tan 7'' \log_e \frac{1600}{775} \text{ miles} = 0{\cdot}0346 \times \frac{0{\cdot}3148}{0{\cdot}414}$$

$$= 0{\cdot}025 \text{ mile} = 132 \text{ feet}.$$

Ex. 2. As the distance from the line increases the force will vary more as the inverse *square*. Suppose that to the distance 1020 miles it varies as the inverse distance, and beyond that as the inverse square. For the first we must integrate as above: thus

$$0{\cdot}0346 \log_e \frac{1020}{775} = 0{\cdot}0346\,\frac{0{\cdot}1193}{0{\cdot}434} = 0{\cdot}0095 \text{ mile} = 50 \text{ feet}.$$

For the more southern part $U = -g \tan 7'' (1020 \div u)^2$, and the rise of the level

$$= 1020 \tan 7'' \left(\frac{1020}{1020} - \frac{1020}{1600}\right) = 0{\cdot}0346 \times \frac{29}{80} = 0{\cdot}01254 \text{ mile} = 66 \text{ feet.}$$

The sum of these is 116 feet, and is somewhat less than the result before obtained. We shall not be above the mark, therefore, in taking the latter.

Ex. 3. If u be the distance, in linear degrees, of the parallel of any place on the west coast of Hindostan from that of Cape Comorin, then the force acting towards the north at any point of that coast, arising from the deficiency of matter in the Ocean, may be approximately represented by the following formula (see Art. 62, Ex. 2):

$$(0{\cdot}000095556839 - 0{\cdot}000002836162\, u + 0{\cdot}000000004072 u^2)\, g.$$

Hence at this place the sea-level would be higher than at Cape Comorin, in consequence of this cause, by

$$0{\cdot}000095556839 u - 0{\cdot}000001418081 u^2 + 0{\cdot}000000001357 u^3.$$

Karachi is about 17° north of Cape Comorin. Hence from this cause, the sea would be higher at Karachi than at Cape Comorin by 0·00122 of a linear degree = 0·8489 mile = 448 feet, if there were no other cause in operation to counteract it.

Ex. 4. To find how much higher the sea-level stands on the shores of Great Britain than it would, if the Ocean in the New Zealand hemisphere were to become land, all other things remaining as at present.

If a great circle be drawn upon the earth as an equator, New Zealand and Great Britain (which are nearly in each other's antipodes) being its poles, the New Zealand hemisphere is nearly all water. We must find the effect of the deficiency of matter in this ocean hemisphere in producing horizontal local attraction in the opposite hemisphere.

(1) We will suppose that this effect is the same as if the New Zealand ocean were of the form of a hemi-spheroidal meniscus, of thickness h at New Zealand: then by Art. 62,

Ex. 3, the horizontal attraction at a place in the Great Britain hemisphere at a distance θ° from New Zealand ($= W$)

$$= (0{\cdot}1446 \sin\theta + 0{\cdot}0958 \sin 2\theta + 0{\cdot}0244 \sin 3\theta) \frac{h}{a} \frac{\rho}{2{\cdot}75} g,$$

ρ being the deficiency of density in the ocean.

Hence, supposing that this place is connected by a canal (as is the case in the North and South Atlantic Ocean) with the New Zealand hemisphere, the consequent elevation of the sea-level there is

$$= \int_{\frac{\pi}{2}}^{\theta} \frac{W}{g}\, d.a\theta = \frac{\rho}{2{\cdot}75} h\, (-0{\cdot}1446 \cos\theta - 0{\cdot}0479 \cos 2\theta - 0{\cdot}0081 \cos 3\theta - 0{\cdot}0479).$$

The density of sea water $= 1{\cdot}028$: hence

$$\rho = 2{\cdot}75 - 1{\cdot}028 = 1{\cdot}72,$$

and the elevation of the sea-level

$$= -0{\cdot}626\, h\, (0{\cdot}0479 + 0{\cdot}1446 \cos\theta + 0{\cdot}0479 \cos 2\theta + 0{\cdot}0081 \cos 3\theta)$$

$$= 0{\cdot}626\, h\, (0{\cdot}1527 - 0{\cdot}0958) \text{ at Great Britain}$$

$$= 0{\cdot}626 \times 0{\cdot}0569\, h = 0{\cdot}036188\, h \text{ mile}$$

$= 382$ feet, if $h = 2$ miles.

(2) Suppose that the ocean in the New Zealand hemisphere is considered to be of the form of a meniscus, the thickness at the pole being zero, and at the edge h. Then by Art. 62, Ex. 5, the elevation of the sea-level

$$= -0{\cdot}626\, (2{\cdot}0608 \cos\theta + 0{\cdot}9442 \cos 2\theta + 0{\cdot}2454 \cos 3\theta + 0{\cdot}9442)\, h$$

$$= 0{\cdot}626\, (2{\cdot}3042 - 1{\cdot}8884)\, h = 0{\cdot}626 \times 0{\cdot}4158\, h$$

$= 0{\cdot}2603\, h = 2748$ feet, if $h = 2$ miles.

(3) Suppose that the ocean is regarded as uniformly deep. Then by Art. 63, Ex. 4, the elevation of the sea-level

$$= -0{\cdot}626\ (2{\cdot}2054\cos\theta + 0{\cdot}9921\cos 2\theta + 0{\cdot}2535\cos 3\theta + 0{\cdot}9921)\ h$$

$$= 0{\cdot}626\ (2{\cdot}4586 - 1{\cdot}9842)\ h = 0{\cdot}626 \times 0{\cdot}4747\ h$$

$$= 0{\cdot}2972\ h = 1570 \text{ feet, if } h = 1 \text{ mile.}$$

The average of these three results is

$$= \frac{1}{3}(382 + 2748 + 1570) = 1567 \text{ feet.}$$

160. There is no doubt that the solid parts of the earth's crust beneath the Pacific Ocean must be denser than in the corresponding parts on the opposite side, otherwise the ocean would flow away to the other parts of the earth. (See Art. 138.) The following reasoning will explain this. Suppose the earth to be a sphere. Through any point on it suppose a surface drawn separating a thin portion on the right hand and through the same point a similar surface separating a like portion on the left. The sphere consists, then, of three parts, the middle portion being of a symmetrical form and attracting the point in the direction of the radius, and the two slender slices attracting it equally to the right and left of that radius. If one of these slices became fluid and of less density than the other, its attraction would be overcome by that of the other, and the fluid would be drawn away to the other parts of the sphere. It does not follow that the whole of the fluid would be drawn over. The above process would go on till the surface of the fluid at the circumference of the slice had become so inclined as to be at right angles to the direction of the resultant attraction of the whole mass, solid and fluid. If, however, a narrow channel were cut through this circumference (which would otherwise act as an embankment) the whole of the water would be drawn off.

Now in the case of the earth there is a channel opening a passage from the New Zealand hemisphere into the opposite one, viz. the North and South Atlantic, and yet the ocean remains in that hemisphere. There must, therefore, be some excess of matter in the solid parts of the earth between the Pacific Ocean and the earth's centre which retains the water

in its place. This effect may be produced in an infinite variety of ways; and therefore, without data, it is useless to speculate regarding the arrangement of matter which actually exists in the solid parts below.

CONCLUSION.

SUMMARY OF EVIDENCE THAT THE EARTH WAS ONCE FLUID.

We can hardly conceive stronger evidence than the foregoing pages furnish that the earth was once a fluid mass.

In the first place we have demonstrated (Art. 74) that as a matter of fact, apart from all theory, the earth's mass lies in strata nearly spherical around the centre. Then, assuming that the earth's mass was once fluid, we deduce from the above fact, that the strata are all spheroidal, having a common centre and a common axis (Art. 77): and we derive general expressions (1) for the value of gravity at the surface, (2) for the perturbation of the moon in latitude by the earth's attraction, and equations for calculating (3) the ellipticity of the surface, and (4) the amount of precession in the earth's axis caused by the sun and moon. These four formulæ flowing from the fluid hypothesis we make use of as tests of the truth of the hypothesis.

The first shows, that in passing from one place to another on the earth's surface the differences of the intensity of gravity should vary as the differences of the squares of the sines of the latitude. This is found to be remarkably verified by pendulum experiments (Art. 92). Besides this the value of the ellipticity, deduced by the application of the result of pendulum experiments to this formula, furnishes another confirmation of the fluid hypothesis, for it very nearly coincides with that determined by an actual measurement of the earth's figure by geodesy (Art. 93). But pendulum experiments are used to bear even stronger testimony to the hypothesis. For we have shown, that any re-arrangement of the mass, without altering its total amount, by merely destroying or even chang-

ing the small ellipticity of the inner strata, would produce sensible effects upon the number of beats made by the pendulum in a given time, such as might easily be detected. The close coincidence, then, of the calculated with the observed number of beats bears indubitable testimony to the *bulging out* of the inner strata, all too in the same direction, namely that corresponding with the equators of the common axis. Better evidence cannot be conceived of the mass having been fluid when the inner parts attained their peculiar arrangement; for it is precisely the one which a mass of fluid revolving round an axis would assume. (See Arts. 94—98.) The second formula also gives a value of the ellipticity, when the observed amount of the moon's perturbation is applied to it, which well accords with the ellipticity found by geodesy (Art. 102).

These results have been obtained without making any hypothesis whatever, except the fundamental one—that the earth assumed its mean form from being fluid—which it is our object to test; and they surely must be regarded as establishing the truth of that hypothesis. We are unable to make use of the third and fourth formulæ without assuming some law of density of the strata; for without this the equations are not soluble. The law which Laplace assumed appears to be a very likely one (Arts. 103, 4). It may be said, moreover, that since the strata are all nearly spherical, one law is (within reasonable limits) as good as another for the purpose for which we use it, the constants involved in its expression being so adjusted as to accord with the facts of the earth. This being the case, we think that the testimony which the third and fourth formulæ bear to our fundamental hypothesis is very important. The value of the ellipticity obtained by the third agrees almost exactly with that given by geodesy (Art. 106); and that deduced from the fourth, by using the observed amount of precession, agrees with the value found by the moon's perturbation in latitude (Art. 110).

When the nature and great difficulty of the problem is considered, and the smallness of the quantities on which the results depend, we think that no doubt can remain on any mind which well weighs the evidence here brought together, that the earth acquired its mean form from being in a fluid

state; and that the variations we at present see in its surface, in mountains and oceans, have been induced since the crust began to form. The conclusion at which we have arrived in Art. 132, that the parts of the crust below the more elevated regions are of less density, and the parts beneath the depressed regions in oceans are of greater density than the average portions of the surface, seems to bear additional testimony to the fluid theory. For it shows, that notwithstanding the varied surface, seen at present in mountains and oceans, the amount of matter in a vertical prism drawn down at various places to any given spheroidal stratum is the same although its length varies from place to place as the earth's contour varies. No better explanation of this phenomenon can be conceived than that which the fluid hypothesis furnishes; viz. that these prisms though now of different lengths, were, when the crust was fluid, of the same length; and as their lengths are now various simply from the fact that the surface in solidifying has contracted unequally, of course the amount of matter which they contain is the same in all of them.

FINIS.

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